18.03 Hour Exam I Solutions: March 1, 2006

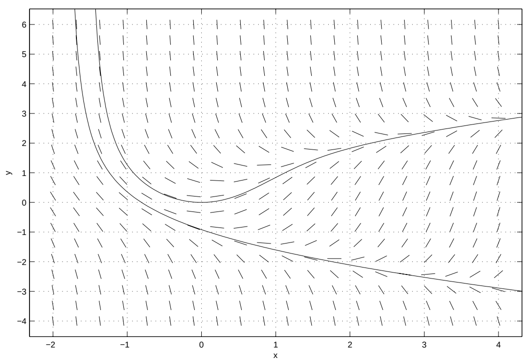
1. **(a)** $\dot{T} = -kT$ off the stove; this has solution $T = Ce^{-kt}$. Thus $C/2 = (1/2)T(0) = T(10\ln 2) = Ce^{-k\cdot 10\ \ln 2} = C(1/2)^{10k}$ and so $10k = 1$ or $k = 1/10$. On the stove the ODE is $\dot{T} = -(1/10)T + 8$.

(b)

k	x_k	y_k	$A_k = x_k + y_k$	$hA_k = A_k$
0	0	0	0	0
1	1	0	1	1
2	2	1		

So $y(2)$ is approximately 1. Since $y_1 = 0 < A_1 = 1$, the vector field has been rising under that segment, and the estimate is too low.

2.



(a) The top curve is the solution through $(0,0)$.

(b) The lower curve is the separatrix: solutions above it grow for x large, solutions below it do not. [In fact, solutions below it reach $-\infty$ in finite time. The separatrix is a solution itself, the only solution which is always falling and which is defined for all large x.]

(c) The graphed solution is trapped by the funnel having the nullcline as its upper fence, so $y(50)$ is very near to $\sqrt{100} = 10$. Since it's approaching from below, the estimate is (very slightly) high.

3. The standard form is $\dot{x} - x = 2te^t$. The homogeneous solution is e^t, so we substitute $x = e^t u$: $\dot{x} = e^t u + e^t \dot{u}$, so $2te^t = \dot{x} - x = e^t \dot{u}$ or $\dot{u} = 2t$. This integrates to $u = t^2 + c$, so $x = t^2 e^t + ce^t$. Since only one solution was asked for we can take $c = 0$ or anything else.

Alternatively, e^{-t} is an integrating factor, and $2t = e^{-t}\cdot 2te^t = e^{-t}(\dot{x} - x) = \dfrac{d}{dt}e^{-t}x$ so $e^{-t}x = \displaystyle\int 2t\,dt = t^2 + c$ or $x = (t^2 + c)e^t$. Again c can be anything.

4. First solve the complex-valued equation $\dot{z} - 2z = 4e^{3it}$. This can be done using integrating factors or variation of parameters, or by trying $z_p = Ae^{3it}$ and solving for A: $A3ie^{3it} - 2Ae^{3it} = 4e^{3it}$ implies $A = 4/(3i - 2)$.

Thus $z_p = \dfrac{4}{-2+3i}e^{3it} = \dfrac{4(-2-3i)}{13}e^{3it}$, whose real part is $x_p = (4/13)(-2\cos(3t) + 3\sin(3t))$.

5. **(a)** The magnitude of i is 1, so the magnitude of each of its cube roots is 1. The argument of i is $\pi/2$, so the argument of one cube root is $\pi/6$. The others differ by $2\pi/3$ and $4\pi/3$ and so are $5\pi/6$ and $9\pi/6 = 3\pi/2$. The last gives $-i$, whose cube is indeed i. The others give $(\sqrt{3} + i)/2$ and $(-\sqrt{3} + i)/2$.

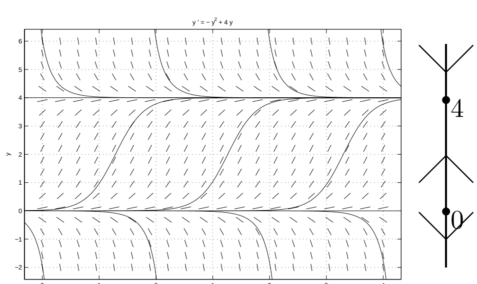
(b) $P = 2\pi/\omega = 2\pi/\pi = 2$.

(c) A is the length of the segment joining $(0,0)$ to $(-2,2)$: $2\sqrt{2}$.

(d) ϕ is the polar angle of $(-2,2)$, which is $3\pi/4$.

(e) $t_0 = (P/2\pi)\phi = (1/\pi)(3\pi/4) = 3/4$.

6. **(a)**



(b) The maximum value of $g(y) = -y^2 + 4y$ is 4 (at $y = 2$), so $\dot{y} = -y^2 + 4y - 4$ has a semi-stable critical point at $y = 2$ and no larger harvest rate leads to any critical points. The largest harvest rate is 4 tons per week.

(c) $\dot{y} = -y^2 + 4y - 3$ has critical points at the roots of $y^2 - 4y + 3$, namely at $y = 1$ and $y = 3$. The critical point at $y = 1$ is unstable and can't be maintained over a long period of time; so the farmer must have $y = 3$.