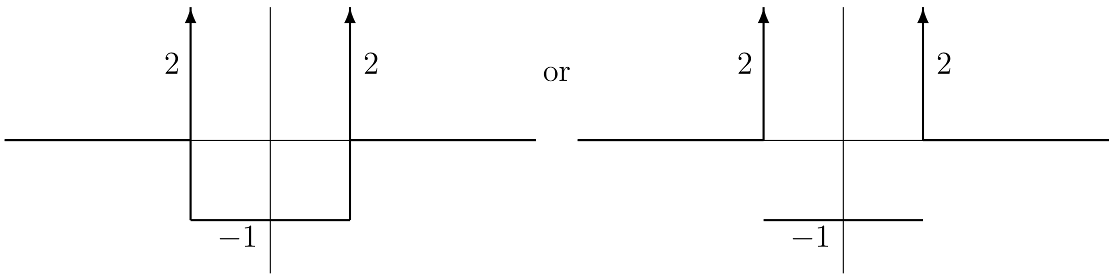
18.03 Hour Exam III Solutions: April 26, 2006

1. (b)



(b) $\delta(t) + \delta\left(t - \frac{1}{12}\right) + \delta\left(t - \frac{2}{12}\right) + \cdots = \sum_{k=0}^{\infty} \delta\left(t - \frac{k}{12}\right)$

2. (a) The period of $\cos(n\pi t)$ is $2\pi/n\pi = 2/n$, and the least common multiple of these, for $n \geq 2$, is 2. This is the minimal period.

(b) $x_p = \frac{\cos(2\pi t)}{2(\omega_n^2 - (2\pi)^2)} - \frac{\cos(3\pi t)}{4(\omega_n^2 - (3\pi)^2)} + \frac{\cos(4\pi t)}{8(\omega_n^2 - (4\pi)^2)} - \cdots = \sum_{k=2}^{\infty} (-1)^k \frac{\cos(k\pi t)}{2^{k-1}(\omega_n^2 - (k\pi)^2)}$

(c) There is resonance when $\omega_n = k\pi$ for $k = 2, 3, 4, \ldots$.

(d) $f(t)$ and $\cos(\pi t)$ are both even, so $f(t)\cos(\pi t)$ is even. Thus $\int_0^1 f(t)\cos(\pi t)\, dt = \frac{1}{2}\int_{-1}^1 f(t)\cos(\pi t)\, dt$, and this is zero either because it computes the Fourier coefficient of $\cos(\pi t)$ in $f(t)$, which is zero, or by the orthogonality properties of the cosine functions, as written in the information page attached to the exam.

3. (a) The denominator $s^2 + 4s + 104 = (s+2)^2 + 100$ [the value announced as a correction in the exam room] vanishes at $s = -2 \pm 10i$, so this is where the poles of $W(s)$ are.

(b) The poles are relatively near to the imaginary axis, so $|W(i\omega)|$ probably has near-resonant peaks near $\omega = \pm 10$. [The gain is an even function of ω, so sketching it for $\omega > 0$ suffices.] The slope becomes zero when $\omega = 0$ (because the function is even), and the graph falls off to zero as $\omega \to \infty$.

(c) $x = u(t-a)e^{I(t-a)} = \begin{cases} 0 & \text{for } t < a \\ e^{I(t-a)} & \text{for } t > a \end{cases}$

4. (a) $(s^2X - (-1)s - 2) + 4(sX - (-1)) + 5X = \frac{2}{s^2+4}$ or $X = \frac{(-s-2) + (2/(s^2+4))}{s^2+4s+5}$

(b) $\frac{s}{s^2+4s+5} = \frac{(s+2)-2}{(s+2)^2+1}$ is the Laplace transform of $e^{-2t}(\cos t - 2\sin t)$

5. (a) $x = w(t) * f(t) = \int_0^t w(t-\tau)f(\tau)\, d\tau = \int_0^t (t-\tau)e^{-(t-\tau)}f(\tau)\, d\tau$. This can also be written $\int_0^t \tau e^{-\tau} f(t-\tau)\, d\tau$.

(b) $W(s) = \mathcal{L}^{-1}[te^{-t}] = -\frac{d}{ds}\frac{1}{s+1} = \frac{1}{(s+1)^2}$, so $W(1) = 1/4$ and $x_p = W(1)e^t = (1/4)e^t$.