18.03: Two practice final exams

Operator Formulas

• Exponential Response Formula: $x_p = Ae^{rt}/p(r)$ solves $p(D)x = Ae^{rt}$ provided $p(r) \neq 0$.

• Resonant Response Formula: If $p(r) = 0$ then $x_p = Ate^{rt}/p'(r)$ solves $p(D)x = Ae^{rt}$ provided $p'(r) \neq 0$.

• Exponential Shift Law: $p(D)(e^{rt}u) = e^{rt}p(D + rI)u$.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for Re $s >> 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s - a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0+)$

$\mathcal{L}[f''(t)] = s^2F(s) - sf(0+) - f'(0+)$

if we ignore singularities in derivatives at $t = 0$.

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(\tau)g(t - \tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = \frac{1}{p(s)}$, $w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2 + \omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}[u_a(t)] = \frac{e^{-as}}{s} \qquad \mathcal{L}[\delta_a(t)] = e^{-as}$$

where $u(t)$ is the unit step function $u(t) = 1$ for $t > 0$, $u(t) = 0$ for $t < 0$.

Fourier series

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + \cdots$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

$$\int_{-\pi}^{\pi}\cos(mt)\cos(nt)\,dt = \int_{-\pi}^{\pi}\sin(mt)\sin(nt)\,dt = 0 \qquad \text{for} \quad m \neq n$$

$$\int_{-\pi}^{\pi}\cos^2(mt)\,dt = \int_{-\pi}^{\pi}\sin^2(mt)\,dt = \pi \qquad \text{for} \quad m > 0$$

If $\mathrm{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$

Variation of parameters

The solution to $\dot{\mathbf{u}} = A\mathbf{u} + \mathbf{q}(t)$ is given by $u = \Phi(t)\int \Phi(t)^{-1}\mathbf{q}(t)\,dt$ where $\Phi(t)$ is any fundamental matrix for A. (In fact this true even if the coefficient matrix $A = A(t)$ is nonconstant. The 1×1 case was studied early on.)

Defective matrix formula

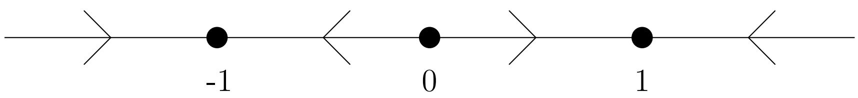
If A is a defective 2×2 matrix with eigenvalue λ_1 and nonzero eigenvector $\mathbf{v}_1$, then you can solve for $\mathbf{w}$ in $(A - \lambda_1 I)\mathbf{w} = \mathbf{v}_1$ and $\mathbf{u} = e^{\lambda_1 t}(t\mathbf{v}_1 + \mathbf{w})$ is a solution to $\dot{\mathbf{u}} = A\mathbf{u}$.

Exam I

1. (a) A vial of pure Kryptonite undergoes radioactive decay, but Lex Luthor keeps it continually resupplied at the rate of $q(t)$ grams per hour. Set up the ODE describing the number $x(t)$ of grams of Kryptonite in the vial. Your answer will involve $q(t)$ and an as yet undetermined decay rate.

(b) Lex actually puts in the Kryptonite in 0.1 gram doses, once an hour on the half hour, starting at $t = 1/2$. Write down an expression for $q(t)$ that models this. (Assume it starts at $t = 0$ and goes on forever.)

2. (a) Sketch the graph of a function $g(x)$ such that the phase line of the autonomous ODE $\dot{x} = g(x)$ looks like this:



(b) Use the Euler method with 3 steps to estimate the value at $x = 0.3$ of the solution to $y' = x + y$ with $y(0) = 1$.

3. (a) Find the general solution to the ODE $t\dot{x} + x = \cos t$.

(b) Suppose that $x_1(t)$ and $x_2(t)$ are solutions of a first order linear ODE, and that $x_1 \neq x_2$. Write down a nonzero solution to the associated homogeneous linear ODE (in terms of x_1 and x_2). Then write down the general solution to the original equation (in terms of x_1 and x_2).

4. (a) Express $(1+i)^{21}$ in the form $a+bi$ with a and b real. (It may be useful to know that $2^{10} = 1024$.)

(b) Write each of the three cube roots of $8i$ in the form $a+bi$ with a and b real.

5. (a) Find a particular real solution to $\ddot{x} + 5x = 4e^{-t}\cos(2t)$.

(b) Find the general real solution to $\ddot{x} + 2\dot{x} + 2x = 2t^2 + 2$.

6. (a) A certain periodic function $f(t)$ has Fourier series given by

$$f(t) = \cos(t) + \frac{\cos(3t)}{9} + \frac{\cos(5t)}{25} + \cdots.$$

What is a periodic solution of the equation $\ddot{x} + \omega_n^2 x = f(t)$ (if one exists)?

(b) What is the Fourier series of the function $g(t)$ which is periodic of period 2 and such that $g(t) = 2$ for $0 < t < 1$ and $g(t) = 0$ for $-1 < t < 0$?

7. (a) For what values of c and k does the LTI operator $L = D^2 + cD + kI$ have unit impulse response given by $w(t) = e^{-2t}\sin(t)$ for $t > 0$?

(b) For this same operator L, write down the convolution integral for the solution to $Lx = e^{-2t}$ with rest initial conditions, and evaluate it.

(c) Find the inverse Laplace transform of $F(s) = \dfrac{4}{(s^2+2s+5)(s+1)}$.

(d) Sketch the pole diagram of $F(s)$.

8. This problem concerns a 2×2 real matrix A whose eigenvalues are 2 and -1, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(a) Write down a fundamental matrix for the vector equation $\dot{\mathbf{u}} = A\mathbf{u}$.

(b) Compute e^{At}.

(c) Find a solution to $\dot{\mathbf{u}} = A\mathbf{u} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

9. Let $A = \begin{bmatrix} a & 4 \\ a & 3 \end{bmatrix}$, and consider the system $\dot{\mathbf{u}} = A\mathbf{u}$. For each of the following conditions, determine all values of a which are such that the system satisfies the condition.

(a) Asymptotically stable

(b) Defective node

(c) Spiral (including centers)

(d) Node

(e) Saddle

(f) There is some constant solution other than $\mathbf{0}$.

10. Two ant species compete with each other for the same food. Each population depresses the growth rate of the other, as well as its own (logistically). Here's a model of this:

$$\begin{aligned} \dot{x} &= (6 - 2x - y)x \\ \dot{y} &= (6 - x - 2y)y. \end{aligned}$$

(a) Find where the vector field is horizontal, where it is vertical, and locate the critical points.

(b) There is one critical point in the upper right quadrant, with positive values of x and y. Find the Jacobian and evaluate it at that critical point.

(c) Sketch the phase portrait of the linearization at this critical point. Plot any eigendirections carefully, and name the type (node, saddle, spiral, . . . ; stable, unstable).

(d) Identify the types of the other critical points, and sketch a phase portrait of this autonomous system in the upper right quadrant.

(e) Can a trajectory leave the upper right quadrant? (Yes or No.)

Exam II

1. Salt water enters a twenty gallon tank at a rate of five gallons per minute, and leaves it at the same rate through a hole in the bottom. A rotor keeps the solution well mixed. Write $x(t)$ for the number of pounds of salt in the tank at time t, and suppose that at $t = 0$ the tank is full of fresh water. Suppose that the concentration of salt in the water being added is $q(t)$ pounds per gallon.

(a) Write down a differential equation that controls $x(t)$. (Measure time in minutes.)

(b) Salt is added to the tank in four sudden discrete packets of half a pound each, once a minute starting at $t = 0$. What is $q(t)$?

2. (a) Use the Euler method with stepsize 1/2 to estimate $y(2)$ if $\frac{dy}{dx} = xy - 1$ and $y(1) = 2$.

(b) Sketch the isoclines for slopes 0 and ± 1 at least for the ODE $\frac{dy}{dx} = x^2 - y$, and use this to sketch the graph of the solution with $y(-1) = 1$ between $x = -2$ and $x = 2$ or more. (Think about what the sign of the slope field is above the isocline for slope zero.)

3. (a) Find the general solution of the ODE $y' - xy = 2e^{x^2/2}\sin(x)$.

(b) Suppose $e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are both solutions to the homogeneous linear system $\dot{\mathbf{u}} = A\mathbf{u}$. What is A? Find a solution of $\dot{\mathbf{u}} - A\mathbf{u} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

4. (a) Express $\cos(\pi t) - \sin(\pi t)$ in the form $A\cos(\omega t - \phi)$.

(b) Express $\frac{1+i}{1-i}$ in the form $a + bi$, a, b real.

(c) Express $1 + i$ as $re^{i\theta}$ with r and θ real and $r \geq 0$.

5. (a) Find a particular solution of $\ddot{x} + 2\dot{x} + x = 4e^t + 5$.

(b) Find the general real solution of $\ddot{x} + 4\dot{x} + 13x = 0$.

(c) Find the amplitude of the sinusoidal solution of $\ddot{x} + 2\dot{x} + x = 2\sin(3t)$.

6. What is the Fourier series for $f(x) = 1 + \cos(x - \pi/4)$?

7. (a) What is the unit impulse response for the operator $D^2 + 2D + 3I$?

(b) The unit impulse response of a certain system is given by $w(t) = \begin{cases} (1/2)e^{-t}\sin(2t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$

Write down the integral expressing the system response to the signal e^{-t} (with rest initial conditions), and evaluate it.

8. (a) Compute e^{At} where $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

(b) Solve $\dot{\mathbf{u}} = A\mathbf{u} + \begin{bmatrix} 2e^t \\ e^t \end{bmatrix}$ with initial condition $\mathbf{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(c) Write down a real matrix A for which $\dot{\mathbf{u}} = A\mathbf{u}$ has as a solution $\mathbf{u} = e^t \begin{bmatrix} \cos t \\ 2\sin t \end{bmatrix}$.

9. As in **8(a)**, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Sketch the phase portrait for $\dot{\mathbf{u}} = A\mathbf{u}$. Mark the eigenlines (the straight solutions) with their corresponding eigenvalues and at least four other trajectories, and name the type of phase portrait you have (saddle, spiral, node, . . . ? stable, unstable?).

10. Write x for the population of bugs (in some convenient units), and y for the population of birds. Birds eat bugs, and the two together satisfy the nonlinear autonomous system

$$\begin{cases} \dot{x} = (2 - x - y)x \\ \dot{y} = (x - 1)y \end{cases}$$

(so that in the absence of birds, the bug population grows logistically, and in the absence of bugs, the birds die out exponentially).

(a) Find all the critical points of this system.

(b) Find the linearization at the critical point with positive x and y coordinates, and sketch the trajectories near that critical point.

(c) Now malathion is introduced in an attempt to reduce the bug population. This reduces the rate of reproduction of both species, so the new system is given by

$$\begin{cases} \dot{x} = (2 - x - y - a)x \\ \dot{y} = (x - 1 - b)y \end{cases}$$

for certain positive constants a, b. What happens to the critical point studied in **(b)**? Is this measure successful in reducing the bug population?

Solutions to Exam I

1. (a) $\dot{x} + kx = q(t)$ where k is the decay rate.

(b) $q(t) = (0.1)(\delta(t - .5) + \delta(t - 1.5) + \delta(t - 2.5) + \cdots)$

2. The $g(x)$ is positive for $x < -1$ and $0 < x < 1$, negative for $-1 < x < 0$ and $x > 1$, and zero at $x = -1$, 0, and 1.

(b) $h = .1$.

k	x_k	y_k	$A_k = x_k + y_k$	hA_k
0	0	1	1	.1
1	.1	1.1	1.2	.12
2	.2	1.22	1.42	.142
3	.3	1.362		

.

3. (a) The equation is $\frac{d}{dt}tx = \cos t$, so $tx = \int \cos t\, dt = \sin t + c$ and $x = \frac{\sin t + c}{t}$.

(b) The difference $x_2 - x_1$ must be a solution to the homogeneous equation, whose general solution is thus $c(x_2 - x_1)$. The general solution of the original equation is then $x_1 + c(x_2 - x_1)$.

4. (a) $|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \pi/4$. Therefore $|(1 + i)^{21}| = 2^{21/2} = 1024\sqrt{2}$ and $\arg((1 + i)^{21}) = 21\pi/4$ or $5\pi/4$. The complex number with these polar coordinates is $-1024(1 + i)$.

(b) $|8i| = 8$ and $\arg(8i) = \pi/2$, so the cube roots have magnitude 2 and arguments $\pi/6$, $5\pi/6$, and $9\pi/6 = 3\pi/2$. These numbers are $(\sqrt{3} + i)$, $(-\sqrt{3} + i)$, and $-2i$.

5. (a) This is the real part of $\ddot{z} + 5z = 4e^{(-1+2i)t}$. Since $p(s) = s^2 + 5$, $p(-1 + 2i) = (-1 + 2i)^2 + 5 = 1 - 4i - 4 + 5 = 2 - 4i$, so the Exponential Response Formula gives $z_p = \frac{4e^{(-1+2i)t}}{2 - 4i}$.

$x_p = \text{Re } z_p = \text{Re } \frac{2(1 + 2i)}{5}e^{-t}(\cos(2t) + i\sin(2t)) = \frac{2}{5}e^{-t}(\cos(2t) - 2\sin(2t))$.

(b) Undertermined coefficients:

2] x_p	$=$	at^2	$+$	bt	$+$	c
2] $\dot{x}_p$	$=$			$2at$	$+$	b
1] $\ddot{x}_p$	$=$					$2a$
$2t^2 + 2$	$=$	$2at^2$	$+$	$(2b + 4a)t$	$+$	$2c + 2b + 2a$

so $a = 1$, $b = -2$, and $c = 2$: $x_p = t^2 - 2t + 2$. The characteristic polynomial $s^2 + 2s + 2 = (s + 1)^2 + 1$ has roots $-1 \pm i$, so basic homogeneous solutions are $e^{-t}\cos t$ and $e^{-t}\sin t$. The general solution is thus $x_p = t^2 - 2t + 2 + e^{-t}(a\cos t + b\sin t)$.

6. (a) $x_p = \frac{\cos(t)}{\omega_n^2 - 1} + \frac{\cos(3t)}{9(\omega_n^2 - 9)} + \frac{\cos(5t)}{25(\omega_n^2 - 25)} + \cdots$.

(b) $g(t) = 1 + \text{sq}(\pi t) = 1 + \frac{4}{\pi}\left(\sin(\pi t) + \frac{\sin(3\pi t)}{3} + \frac{\sin(5\pi t)}{5} + \cdots\right)$.

7. (a) The unit impulse response is a homogeneous solution, and to get $e^{-2t}\sin(t)$ the roots of the characteristic polynomial must be $-2 \pm i$. The sum of the roots is -4 and the product is 5, so the characteristic polynomial is $s^2 + 4s + 5$: $c = 4$, $k = 5$.

(b) $x = e^{-2t}\sin(t) * e^{-2t} = e^{-2t} * e^{-2t}\sin(t) = \int_0^t e^{-2(t-\tau)}e^{-2\tau}\sin(\tau)\, d\tau$

$= e^{-2t}\int_0^t \sin(\tau)\, d\tau = e^{-2t}(-\cos(t) + 1)$.

(c) $F(s) = \dfrac{4}{(s^2+2s+5)(s+1)} = \dfrac{a(s+1)+b}{(s+1)^2+4} + \dfrac{c}{s+1}$. Multiply through by the first denominator and set $s=-1+2i$: $\dfrac{4}{(-1+2i)+1} = a(2i)+b$, or $-2i = 2ai+b$ so $a=-1, b=0$. Multiply through by the second denominator and set $s=-1$: $\dfrac{4}{1-2+5} = c$ or $c=1$. Thus $F(s) = \dfrac{-(s+1)}{(s+1)^2+4} + \dfrac{1}{s+1}$. Write $G(s) = \dfrac{-s}{s^2+4} + \dfrac{1}{s}$, so $F(s) = G(s+1)$. $g(t) = -\cos(2t)+1$, and $f(t) = e^{-t}(-\cos(2t)+1)$.

(d) There are poles at $s=-1\pm 2i$ and at $s=-1$.

8. **(a)** $\Phi(t) = \begin{bmatrix} e^{2t} & 2e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix}$.

(b) $\Phi(0) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$; $\Phi(0)^{-1} = \dfrac{1}{-3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$;
$e^{At} = \Phi(t)\Phi(0)^{-1} = \dfrac{1}{3}\begin{bmatrix} -e^{2t}+4e^{-t} & 2e^{2t}-2e^{-t} \\ -2e^{2t}+2e^{-t} & 4e^{2t}-e^{-t} \end{bmatrix}$

(c) Look for a constant solution: $\mathbf{u} = -A^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. To find A notice that $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = -\begin{bmatrix} 2 \\ 1 \end{bmatrix}$; or, putting these equations side by side in a matrix, $A\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & -1 \end{bmatrix}$. Thus $A = \begin{bmatrix} 2 & -2 \\ 4 & -1 \end{bmatrix}\dfrac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}$, and $A^{-1} = \begin{bmatrix} -3/2 & 1 \\ -1 & 1 \end{bmatrix}$. The constant solution is thus $-\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

9. $p_A(\lambda) = \lambda^2 - (a+3)\lambda - a$. The roots are $\dfrac{(a+3)\pm\sqrt{(a+3)^2+4a}}{2}$. They are repeated when $a^2+10a+9=0$, i.e. when $a=-1$ or $a=-9$. For $-9<a<-1$ they are non-real. The real parts of the eigenvalues are negative when tr$A<0$ and det$A>0$, i.e. $a+3<0$ and $-a>0$ or $a<-3$.

(a) $a<-3$. **(b)** $a=-1,-9$. **(c)** $-9<a<-1$. **(d)** $a<-9$ and $-1<a<0$. **(e)** $0<a$. **(f)** $a=0$.

10. **(a)** The vector field is vertical where $\dot{x}=0$: $x=0$ or $y=6-2x$. It is horizontal where $\dot{y}=0$: $y=0$ or $y=3-(x/2)$. There are critical points at $(0,0),(0,3),(3,0),(2,2)$.

(b) $J(x,y) = \begin{bmatrix} 6-4x-y & -x \\ -y & 6-x-4y \end{bmatrix}$ so $J(2.2) = \begin{bmatrix} -4 & -2 \\ -2 & -4 \end{bmatrix}$.

(c) With $A=J(2,2)$, $p_A(\lambda) = \lambda^2+8\lambda+12$ so the eigenvalues are -2 and -6. A nonzero eigenvector for -2 is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and for -6 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This gives a stable node, in which the non-ray trajectories become tangent to the eigenline through $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as $t\to\infty$.

(d) $J(0,0) = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$ gives an unstable star.

$J(0,3) = \begin{bmatrix} 3 & 0 \\ -3 & -6 \end{bmatrix}$ has eigenvalues 3 and -6 and gives a saddle. A nonzero eigenvector for $\lambda = 3$ is $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$, for -6 is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$J(3,0) = \begin{bmatrix} -6 & -3 \\ 0 & 3 \end{bmatrix}$ has eigenvalues -6 and 3 and gives a saddle. A nonzero eigenvector for $\lambda = -6$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and for 3 is $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

(e) No.

Solutions to Exam II

1. (a) $\dot{x} + (1/4)x = 5q(t)$

(b) $q(t) = (1/2)(\delta(t) + \delta(t-1) + \delta(t-2) + \delta(t-3) + \cdots)$

2. (a) $h = .5$.

k	x_k	y_k	$A_k = x_k + y_k$	hA_k
0	1	2	1	.5
1	1.5	2.5	2.75	1.375
2	2	3.875		

.

(b) Not shown.

3. (a) $y = e^{x^2/2}(C - 2\cos x)$.

(b) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $\mathbf{u} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

4. (a) $\omega = \pi t \qquad A = \sqrt{2} \qquad \phi = -\pi/4$

(b) i.

(c) $1 + i = \sqrt{2}e^{i\pi/4}$.

5. (a) $e^t + 5$.

(b) $e^{-2t}(a\cos(3t) + b\sin(3t))$.

(c) 1/5

6. $1 + \cos(\pi/4)\cos x + \sin(\pi/4)\sin x$.

7. (a) $(1/sqrt2)e^{-t}\sin(\sqrt{2}t)$.

(b) $w(t) * e^{-t} = \int_0^t (1/2)e^{-u}\sin(2u)e^{-(t-u)}du = (1/4)e^{-t}(1 - \cos(2t))$.

8. (a) $\begin{bmatrix} e^t & -e^t + e^{2t} \\ 0 & e^{2t} \end{bmatrix}$.

(b) $\begin{bmatrix} (t-1)e^t + e^{2t} \\ -e^t + e^{2t} \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & -1/2 \\ 2 & 1 \end{bmatrix}$.

9. unstable node.

10. (a) $(0,0)$, $(2,0)$, $(1,1)$.

(b) $\dot{u} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} u$ a stable spiral (counterclockwise).

(c) The new critical point is $(1+b, 1-a+b)$. It is still stable. Clearly there are more bugs.