18.03 Problem Set 2 Solutions: Part II

4. **(a)** x decays exponentially to zero. y rises to a maximum and then falls off exponentially to zero. z increases monotonically and exponentially approaches the limiting value of one mole.

(b) First recall the relationship between the decay constant and the half life: e^{-lt} solves $\dot{x} = -lx$, and $e^{-lt_L} = 0.5$, so $lt_L = \ln 2$ or $l = \ln 2/t_L$. So the requested notation is well chosen: the decay constant for Kriptonite is k and for Luthorium is l.

Differential equations: $\dot{x} + lx = 0$. $\dot{y} + ky = lx$. $\dot{z} = ky$.

(c) With $x(0) = 1$, the solution is $x = e^{-lt}$. Then $\dot{y} + ky = le^{-lt}$. This is an inhomogeneous linear ODE. The homogeneous solution is $y_h = e^{-kt}$. Substitute $y = e^{-kt}u$:

$$le^{-lt} = \frac{d}{dt}(e^{-kt}u) + ke^{-kt}u = e^{-kt}\dot{u} - ke^{-kt}u + ke^{-kt}u = e^{-kt}\dot{u}\,.$$

Thus $\dot{u} = le^{kt}e^{-lt} = le^{(k-l)t}$, so $u = (l/(k-l))e^{(k-l)t} + c$ and $y = e^{-kt}u = (l/(k-l))e^{-lt} + ce^{-kt}$. The initial condition $y(0) = 0$ forces $c = -l/(k-l)$, so $y = (l/(k-l))(e^{-lt} - e^{-kt})$. Finally, $z = k\int y\,dt = (kl/(k-l))(e^{-lt}/(-l) - e^{-kt}/(-k)) + c$. $z(0) = 0$ gives $c = 1$, so $z = \dfrac{1}{k-l}\left(le^{-kt} - ke^{-lt}\right) + 1$.

(d) If y reaches a maximum at $t = t_m$, say, then $\dot{y}(t_m) = 0$. This happens when the derivatives of the two exponentials in the formula for y cancel: $le^{-lt_m} = ke^{-kt_m}$. Multiply by e^{kt_m}: $e^{(k-l)t_m} = k/l$, or $(k-l)t_m = \ln k - \ln l$, or $t_m = (\ln k - \ln l)/(k-l)$.

5. **(a)** $\dfrac{2}{1-i} = \dfrac{2}{1-i} \cdot \dfrac{1+i}{1+i} = 1+i$. The argument is $\pi/4$ and the magnitude is $\sqrt{2}$ so we get $\sqrt{2}e^{i\pi/4}$.

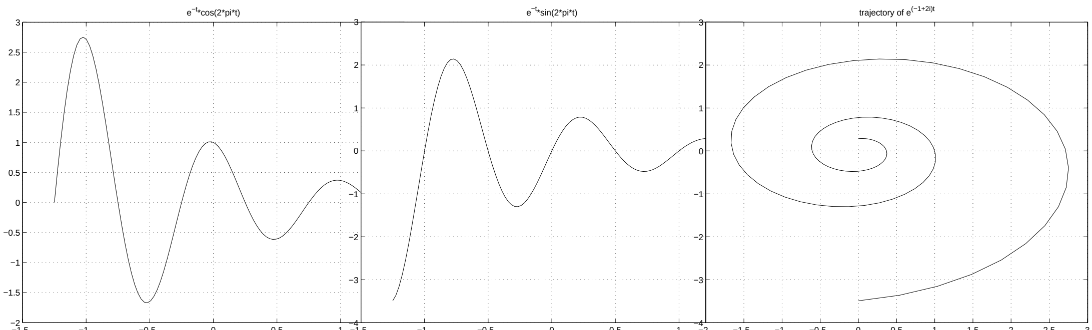
(b) The modulus is e and the argument is $\pi/3$. $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$, so the real part is $e/2$ and the imaginary party is $\sqrt{3}e/2$.

(c) The modulus of a fourth root of -1 must be 1, since it is a positive real number whose fourth power is $|-1| = 1$. The argument must be one quarter of an argument of -1. The argument of -1 is only defined up to adding integer multiples of 2π, so when I take a quarter of it I get a number which is only defined up to adding integer multiples of $\pi/2$. One argument of -1 is π, so the arguments of the fourth roots are given by $\pi/4$ plus integer multiples of $\pi/2$: $\pm\pi/4, \pm 3\pi/4$. These have rectangular descriptions: $(\pm 1 \pm i)/\sqrt{2}$.

(d) The modulus of e^{a+bi} is e^a, and $|1+i| = \sqrt{2}$, so if $e^{a+bi} = 1+i$ then $e^a = \sqrt{2}$ or $a = (\ln 2)/2$. The argument of e^{a+bi} is b, and the argument of $1+i$ is $\pi/4$. But the argument is only defined up to adding integer multiples of 2π, so b can be $(8k+1)(\pi/4)$ for any integer k.

6. **(a)** $e^{4it} = \cos(4t) + i\sin(4t)$. On the other hand, $(e^{it})^4 = (\cos t + i\sin t)^4$. The imaginary part of this power has contributions whenever the sine term is raised to an odd power: it is $4\cos^3 t\sin t - 4\cos t\sin^3 t$.

(b) $e^{-t}\cos(2\pi t) = \operatorname{Re} e^{zt}$ for $z = -1 + 2\pi i$. $\operatorname{Im} e^{zt} = e^{-t}\sin(2\pi t)$ has the middle graph below. The curve in $\mathbb{C}$ parametrized by e^{zt} looks like the right hand graph.



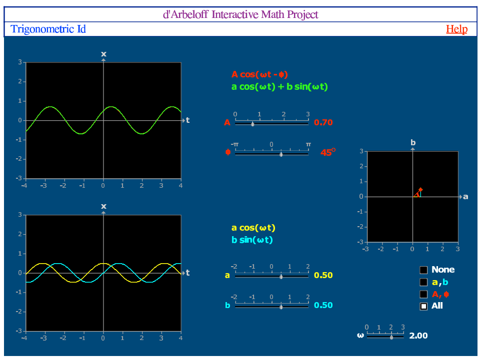
(c) $a = 0$, $b \neq 0$: $|e^{(a+bi)t} = e^{at}$ must be constant to get a circle, so $a = 0$; while the argument of $e^{(a+bi)t}$ takes on all values as t varies as long as $b \neq 0$.

(d) $b = 0$, $a \neq 0$: The curve will rotate if b is not zero; and its distance from 0 won't change if $a = 0$. The only possible ray is the positive real axis.

(e) $a < 0$; $|e^{(a+bi)t}| = e^{at}$ converges to zero as $t \to \infty$ exactly when $a < 0$.

(f) $a > 0$ and $b > 0$: $|e^{(a+bi)t}| = e^{at}$ is increasing exactly when $a > 0$, and the angle bt is increasing exactly when $b > 0$.

7. (a) $\dfrac{e^{it/2}}{1+i} = \dfrac{1-i}{2}(\cos(t/2) + i\sin(t/2))$ so $\omega = 1/2$, $a = 1/2$, and $b = 1/2$. [Another correct answer is $\omega = -1/2$, $a = 1/2$, $b = -1/2$; but normally we expect $\omega \geq 0$ since one can always arrange this.] By SN A, ϕ are the polar coordinates of the point in the plane with rectangular cooridinates (a, b). So $A = 1/\sqrt{2}$ and $\phi = \pi/4$.



(b) $x(t + \Delta t) \simeq x(t) + k(y(t) - x(t))\Delta t$ by the same reasoning as the rootbeer cooler model in Lecture. This leads to $\dot{x} + kx = ky$. The system is the canal, whose characteristics are captured by the coupling constant k. The input signal is the elevation of the ocean, y, or rather k times that. The output signal is the water height in the bay, x.

(c) The period is 4π, so the circular frequency $\omega = 2\pi/P = 1/2$. The equation we are looking at is $\dot{x} + kx = k\cos(\omega t)$.

(d) To answer this we have to find the periodic solution to the equation. Let's do this by using the complex exponential. The equation is the real part of $\dot{z} + kz = e^{i\omega t}$. By the ERF, this has solution given by $z_p = ke^{i\omega t}/(i\omega + k)$. This function parametrizes a circle of radius $k/|i\omega + k|$. Its real part is the periodic solution of our original equation, and it has maximal value $k/|i\omega + k|$. Setting this equal to $1/\sqrt{2}$ gives $\omega^2 + k^2 = 2k^2$ or $k = \omega$. Now remember that $\omega = 1/2$, so $k = 1/2$.

We know that the amplitude of the steady state solution is $A = 1/\sqrt{2}$. To find the phase lag ϕ, we remember that the the expression $x_p = A\cos(\omega t - \phi)$ arises as the real part of $z_p = ke^{i\omega t}/(i\omega + k) = (1/2)e^{it/2}/(i/2 + 1/2) = e^{it/2}/(1 + i)$. But it's also the real part of $Ae^{i(\omega t - \phi)}$, so will try to write $z_p = e^{it/2}/(1 + i)$ in the form $Ae^{i(\omega t - \phi)}$. To do this, write $1/(1 + i) = (1 - i)/2$ in polar form: $(1 - i)/2 = (1/\sqrt{2})e^{-\pi i/4}$. Substituting, $z_p = (1/\sqrt{2})e^{-\pi i/4}e^{it/2} = (1/\sqrt{2})e^{i(t/2 - \pi/4)}$. The result is that $A = 1/\sqrt{2}$ (as we know), and $\phi = \pi/4$. This checks with the Mathlet!