18.03 Problem Set 6 Solutions: Part II

Each problem is worth 16 points, spread across Parts I and II. Part I values: 22: 7C-1, 3 pts; 7C-2, 1 pt. 23: 4 pts. 24: 4 pts.

22. (a) [2] $1 = \mathrm{sq}(\pi/4) = \frac{4}{\pi}\sum_{k \text{ odd}} \frac{\sin(k\pi/4)}{k}$.

k	$\sin(k\pi/4)$
1	$1/\sqrt{2}$
3	$1/\sqrt{2}$
5	$-1/\sqrt{2}$
7	$-1/\sqrt{2}$
$\vdots$	$\vdots$

so

$$1 = \frac{4}{\pi}\left(1/\sqrt{2} + \frac{1/\sqrt{2}}{3} - \frac{1/\sqrt{2}}{5} - \frac{1/\sqrt{2}}{7} + + - - \cdots\right) \text{ or } \boxed{1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + + - - \cdots = \frac{\sqrt{2}\pi}{4}}.$$

(b) (i) [2] Method I: For $n > 0$, $a_n = \frac{2}{2}\int_0^2 t\cos\left(\frac{n\pi t}{2}\right) dt$. Integrate by parts: $u = t$, $dv = \cos\left(\frac{n\pi t}{2}\right) dt$, $du = dt$, $v = (2/n\pi)\sin\left(\frac{n\pi t}{2}\right)$:

$$a_n = \frac{2t}{n\pi}\sin\left(\frac{n\pi t}{2}\right)\Big|_0^2 - \int_0^2 \frac{2}{n\pi}\sin\left(\frac{n\pi t}{2}\right) dt = \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi t}{2}\right)\Big|_0^2$$

The values of the cosine alternate between -1 (for n odd) and $+1$ (for n even), so $a_n = -8/n^2\pi^2$ for n odd and $a_n = 0$ for n even. Of course a_0 is twice the average value of $f(t)$, which is 0: $f(t) = -\frac{8}{\pi^2}\left(\cos(\pi t/2) + \frac{\cos(3\pi t/2)}{9} + \cdots\right)$.

Alternatively, in Lecture 22 we computed that the Fourier series of the even periodic function $g(t)$ of period 2π which is t between 0 and π is $\frac{\pi}{2} - \frac{4}{\pi}\left(\cos(t) + \frac{\cos(3t)}{9} + \cdots\right)$. Then $f(t) = \frac{2}{\pi}g\left(\frac{\pi t}{2}\right) - 1 = -\frac{8}{\pi^2}\left(\cos(\pi t/2) + \frac{\cos(3\pi t/2)}{9} + \cdots\right)$.

A periodic solution to $\ddot{x} + \omega_n^2 x = f(t)$ is thus given by

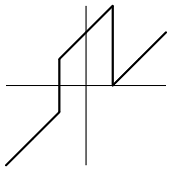
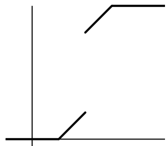
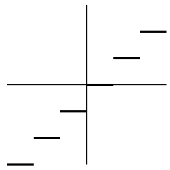
$$x_p = -\frac{8}{\pi^2}\left(\frac{\cos(\pi t/2)}{\omega_n^2 - (\pi/2)^2} + \frac{\cos(3\pi t/2)}{9(\omega_n^2 - (3\pi/2)^2)} + \frac{\cos(5\pi t/2)}{25(\omega_n^2 - (5\pi/2)^2)} + \cdots\right).$$

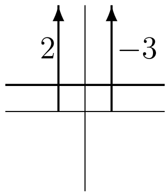
(ii) [2] When $\omega_n = k\pi/2$ for k an odd integer.

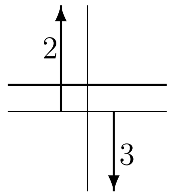
(iii) [2] The smallest such (positive) value is $\pi/2$. For ω_n just less than this, the term $-\frac{8}{\pi^2}\frac{\cos(\pi t/2)}{\omega_n^2 - (\pi/2)^2}$ dominates the sum. This is a very large multiple of $\cos(\pi t/2)$, in phase.

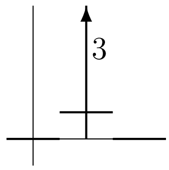
(iv) [2] When it exists (i.e. when ω_n is not an odd multiple of $\pi/2$), the particular solution x_p is periodic of minimal period $P = (2\pi)/(\pi/2) = 4$. In that case, the general solution is $x_p + x_h$, where $x_h = a\cos(\omega_n t) + b\sin(\omega_n t)$. If some integral multiple of the period of x_p is also an integral multiple of the period of x_h, then that common number is a period for $x_p + x_h$. The period of x_h is $2\pi/\omega_n$, so what is required is that there are integers k and l such that $4k = (2\pi/\omega_n)l$. This is the same as requiring that ω_n should be a rational multiple of π. [This is a tricky problem!]

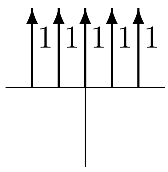
(v) [0] This problem is even trickier (and trickier than I had intended). My point was that in the case just studied, *all* solutions are periodic.

23. (a) [6] (i)  (ii)  (iii) 

(b) [6] (i) $f'(t) = 1+2\delta(t+1)-3\delta(t-1)$. 

or 

(ii) $g'(t) = u(t-1) - u(t-3) + 3\delta(t-2)$. 

(iii) $h'(t) = \displaystyle\sum_{k=-\infty}^{\infty} \delta(t-k)$. 

24. (i) [2] For $t > 0$ the unit step response satisfies $2\dot{x} + kx = 1$ with initial condition $x(0) = 0$. $x_p = 1/k$, $x_h = ce^{-kt/2}$, $x = (1/k)(1 - e^{-kt/2})$. The unit step response is $v(t) = (1/k)(1 - e^{-kt/2})$ for $t > 0$, $v(t) = 0$ for $t < 0$.

[2] For $t > 0$ the unit step response satisfies $\ddot{x} + 2\dot{x} + 5x = 1$ with initial condition $x(0) = \dot{x}(0) = 0$. $x_p = 1/5$, $x_h = e^{-t}(a\cos(2t) + b\sin(2t))$. With $x = x_p + x_h$, $x(0) = 0$ implies that $a = -1/5$. Then $\dot{x} = e^{-t}((-a+2b)\cos(2t) + (-2a-b)\sin(2t))$ so $0 = \dot{x}(0) = (-a+2b)$, which implies $b = -1/10$: $v = (1/5) - (1/10)e^{-t}(2\cos(2t) + \sin(2t))$ for $t > 0$, $v = 0$ for $t < 0$.

[2] For $t > 0$ the unit step responses satisfies $\dfrac{d^3x}{dt^3} = 1$, $x(0) = \dot{x}(0) = \ddot{x}(0) = 0$: so $v = t^3/6$ for $t > 0$, $v = 0$ for $t < 0$.

(ii) [6] The unit impulse responses can be obtained directly, or by differentiating the unit step responses. They are: for $t > 0$, $w = (1/2)e^{-kt/2}$; $w = (1/2)e^{-t}\sin(2t)$; $w = t^2/2$.

Graphs omitted from this solution sheet, but count 1 point each. The main points: In the first one, $v(0+) = 0$ and $v(t) \to 1/k$ as $t \to \infty$; and $w(0+) = 1/2$. In the second one, $v(0) = \dot{v}(0) = 0$ and $v(t)$ oscillates around the value $1/5$ and converges to $1/5$ as $t \to \infty$; $w(0+) = 0$, $\dot{w}(0+) = 1$, and $w(t)$ is a damped sinusoid.