18.03 Problem Set 7 Solutions: Part II

Each problem is worth 16 points, spread across Parts I and II. Part I values: 25: 4 points. 26: 5 points; 27: 8 points.

Comment on I.25, EP 4.1: 7-9: The solution key claims that the integral defining these Laplace transforms converge only for $s > 0$ (by which is meant $\mathrm{Re}\,(s) > 0$), but this is wrong: these integrals converge for all s, since for $t > 2$ the functions being integrated are zero so the improper integral certainly has a limit. More about this when we talk about poles.

25. (a) [4] Write out both sides. You have to use some choices of names for the variables of integration. I'll pick u, v, and x, y:

$$((f*g)*h)(t) = \int_0^t (f*g)(t-u)h(u)\,du = \int_0^t \int_0^{t-u} f(t-u-v)g(v)h(u)\,dv\,du$$

$$(f*(g*h))(t) = \int_0^t f(t-x)(g*h)(x)\,dx = \int_0^t \int_0^x f(t-x)g(x-y)h(y)\,dy\,dx\,.$$

Matching what f, g, and h are being applied to suggests the change of variables $u = y$, $v = x - y$. In the first integral, then, replace u by y. In the inside integral, y is fixed, so $dv = dx$ and we can rewrite the first integral as $\displaystyle\int_0^t \int_y^t f(t-x)g(x-y)h(y)\,dx\,dy$. Now we have to reverse the order of integration, and this gives us the second integral.

(b) [1] $w(t) * u(t) = \int_0^t w(t)$ by definition of convolution. On the other hand, $w(t) * q(t)$ is the system response (from rest initial conditions) to the input signal $q(t)$; so this must be the unit step response.

(c) (i) [2] $\sin(0) = 0$ and $\sin'(0) \neq 0$, so we expect a second order system. The weight function is (for $t > 0$) a homogeneous solution, and $x = \sin(t)$ is a solution to $\ddot{x} + x = 0$. The "mass" is 1 here, leading to the correct value $\sin'(0) = 1$.

So we must solve $\ddot{x} + x = \sin(t)$. This is the imaginary part of $\ddot{z} + z = e^{it}$. The characteristic polynomial $s^2 + 1$ has i as a root, so we are in resonance. The Resonant Response Formula then gives $z_p = te^{it}/(2i)$, which has imaginary part $x_p = -(t/2)\cos t$. This has $x(0) = 0$, so let's just check to see what $\dot{x}_p(0)$ is—maybe we have already hit on the solution with rest initial conditions. $\dot{x}_p = (t/2)\sin t - (1/2)\cos t$, and $\dot{x}(0) = -1/2$: so we have to add a homogeneous solution with $x_h(0) = 0$ and $\dot{x}_h(0) = 1/2$. $(1/2)\sin t$ does the trick: the solution is, for $t > 0$, $x = (1/2)(-t\cos t + \sin t)$.

(ii) [2] $\sin(t) * \sin(t) = \displaystyle\int_0^t \sin(t-\tau)\sin(\tau)\,d\tau$. To go on, use the sine difference formula $\sin(t-\tau) = \sin t\cos\tau - \cos t\sin\tau$:

$$\cdots = \sin t \int_0^t \cos\tau\sin\tau\,d\tau - \cos t \int_0^t \sin^2\tau\,d\tau\,.$$

Next $\sin^2\tau = \dfrac{1-\cos(2\tau)}{2}$, so

$$\cdots = \sin t\left[\frac{\sin^2\tau}{2}\right]_0^t - \cos t\left[\frac{\tau}{2}\right]_0^t + \cos t\left[\frac{\sin(2\tau)}{4}\right]_0^t = \frac{\sin^3 t}{2} - (\cos t)\frac{t}{2} + (\cos t)\frac{\sin(2t)}{4}\,.$$

[This expression has two terms not present in the result from (i). But
$\frac{\sin^3 t}{2} + (\cos t)\frac{\sin(2t)}{4} = (\sin t)\frac{1-\cos(2t)}{4} + (\cos t)\frac{\sin(2t)}{4} = \frac{\sin t}{4} + \frac{\cos(t)\sin(2t) - \sin(t)\cos(2t)}{4}$,
and the right numerator is the sine difference formula for $\sin(2t-t)$, so the two terms combine to give $(\sin t)/2$ and we recover the same function as in (i).]

(iii) [2] The solution to $p(D)x = \sin t$ (with rest initial conditions) is given by the convolution $\sin t * \sin t = \int_0^t \sin(t-\tau)\sin(\tau)\,d\tau$. When $t = k\pi$ for k an odd integer), $\sin(t-\tau) = \sin(k\pi - \tau) = \sin\tau$, so the integrand is $\sin^2\tau$ and is always positive. As k increases you are adding up more and more of the humps, and the solution grows. When k is an even integer, $\sin(k\pi-\tau) = -\sin\tau$, so we aren integrating $-\sin^2\tau$; the integral is negative and it grows as k grows. In sum, the flipped weight function comes into synchrony with the signal from time to time, producing large system response.

(iv) [1] From the discussion in (iii), it seems likely that the maxima occur at $t = k\pi$ for k an odd integer. Let's see: in (i) we computed $\dot{x}_p = (t/2)\sin t - (1/2)\cos t$, so $\dot{x} = (t/2)\sin t$. This is zero when t is an integral multiple of π, and the extrema alternate between maxima and minima.

26. (a) [3] $G(s) = \int_0^\infty f(at)e^{-st}\,dt$. Make a change of variables: $u = at$, $du = a\,dt$, $\int_0^\infty f(u)e^{-s(u/a)}(1/a)\,du$. Pull the $1/a$ outside and recognize what is left as $F(s/a)$: so $G(s) = (1/a)F(s/a)$. As a check, with $f(t) = e^t$, $g(t) = e^{at}$, so $F(s) = 1/(s-1)$ and $G(s) = 1/(s-a)$. Well, $\frac{1}{a}F(s/a) = \frac{1}{a((s/a)-1)} = \frac{1}{s-a} = G(s)$.

(b) [3] With $f(t) = u(t)$, $F(s) = 1/s$, so $L[\delta(t)] = sF(s) - f(0+) = s/s - 1 = 0$. The problem is that (iii) is only true if $f'(t)$ means the ordinary derivative rather than the generalized derivative $f(t)$ is continuous for $t > 0$. The ordinary derivative of $u(t)$ is zero, and is true that $L[0] = sF(s) = 1$.

(c) [3] If $f(t)$ is continuous and piecewise differentiable, then the generalized derivative is $f'(t) = (f')_r(t) + f(0+)\delta(t)$, where $(f')_r(t)$ is the ordinary derivative. Thus $sF(s) = L[f'(t)] = L[(f')_r(t) + f(0+)\delta(t)) = L[(f')_r(t)] + f(0+)$, so $L[(f')_r(t)] = sF(s) - f(0+)$.

(d) [2] $f'(t) = \delta(t) - \delta(t-1)$ and $L[\delta(t) - \delta(t-1)] = 1 - e^{-s}$. On the other hand, $F(s) = (1/s) - (e^{-s}/s)$ (using $L[u(t)] = 1/s$ and the s-shift rule), so $sF(s) = 1 - e^{-s}$.

27. [8] Since $L[tf(t)] = -F'(s)$,

$$L[t\cos(\omega t)] = -\frac{d}{ds}\frac{s}{s^2+\omega^2} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2} \quad \text{and} \quad L[t\sin(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2}.$$

Thus $L^{-1}\left[\frac{s}{(s^2+\omega^2)^2}\right] = \frac{t\sin(\omega t)}{2\omega}$. For the other we use $L[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2} = \frac{\omega(s^2+\omega^2)}{(s^2+\omega^2)^2}$ again to cancel the s^2 term from the numerator: $L[\sin(\omega t) - \omega t\cos(\omega t))] = \frac{2\omega^3}{(s^2+\omega^2)^2}$, and $L^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right] = \frac{\sin(\omega t)}{2\omega^3} - \frac{t\cos(\omega t)}{2\omega^2}$.