18.03 Problem Set 8 Solutions: Part II

Each problem is worth 16 points, spread across Parts I and II. Part I values: 28: 2 points. 31: 4 points; 32: 3 points for 4C-1, 1 point for 4C-4, 2 points for 4C-6ab, for a total of 6 points.

28. (a) (i) The roots of the characteristic polynomial are $-1 \pm i$, so the basic real homogeneous solutions are $e^{-t}\cos t$ and $e^{-t}\sin t$.

[1] $f(t) = \delta(t)$: this gives initial conditions $x(0+) = 0$, $\dot{x}(0) = 1$, which give solution $e^{-t}\sin t$.

[1] $f(t) = u(t)$: this gives initial conditions $x(0) = \dot{x}(0) = 0$ and equation $\ddot{x} + 2\dot{x} + 2x = 1$ for $t > 0$. A particular solution (by ERF if you want!) is $x_p = 1/2$. With $x = (1/2) + e^{-t}(a\cos t + b\sin t)$, $0 = x(0) = (1/2) + a$ so $a = -1/2$; then $\dot{x} = e^{-t}((-(-1/2) + b)\cos t + ??\sin t)$, so $0 = \dot{x}(0) = (1/2) + b$ and $b = -1/2$: so $x = (1/2)(1 - e^{-t}(\cos t + \sin t))$.

[2] $f(t) = \cos(2t)$: the corresponding complex exponential ODE is $\ddot{z} + 2\dot{z} + 2z = e^{2it}$. $p(2i) = (2i)^2 + 2(2i) + 2 = -2 + 4i$, so by ERF $z_p = e^{2it}/(-2+4i) = ((-2-4i)/20)(\cos(2t) + i\sin(2t))$. Thus $x_p = \mathrm{Re}\,(z_p) = (-1/10)\cos(2t) + (1/5)\sin(2t)$. $x_p(0) = -1/10$ and $\dot{x}_p(0) = 2/5$, so we want the transient $x_h = e^{-t}(a\cos t + b\sin t)$ to have $x_h(0) = 1/10$ and $\dot{x}_h(0) = -2/5$. This gives $a = 1/10$, and then $\dot{x}_h = e^{-t}((-(1/10) + b)\cos t + ??\sin t)$, so $-2/5 = -(1/10) + b$ or $b = -3/10$: $x = (-1/10)\cos(2t) + (1/5)\sin(2t) + e^{-t}((1/10)\cos t - (3/10)\sin t)$.

(ii) [1] $f(t) = \delta(t)$: $X(s) = W(s) = 1/p(s) = 1/((s+1)^2 + 1)$ has inverse Laplace transform $w(t) = e^{-t}\sin t$.

[1] $f(t) = u(t)$: $X(s) = \dfrac{1}{s((s+1)^2+1)} = \dfrac{a}{s} + \dfrac{b(s+1)+c}{(s+1)^2+1}$. Cover up the s to see $a = 1/2$. Cover up the $(s+1)^2 + 1$ and set $s = -1 + i$ to see $1/(-1+i) = bi + c$ or $b = c = -1/2$: $X(s) = \dfrac{1}{s} - \dfrac{1}{2}\dfrac{(s+1)+1}{(s+1)^2+1}$. Thus $x = (1/2)(1 - e^{-t}(\cos t + \sin t))$.

[2] $f(t) = \cos(2t)$: $X = \dfrac{s}{(s^2+4)((s+1)^2+1)} = \dfrac{as+b}{s^2+4} + \dfrac{c(s+1)+d}{(s+1)^2+1}$. Cover up the $s^2 + 4$ and set $s = 2i$: $\dfrac{2i}{(2i+1)^2+1} = a(2i) + b$. $(2i+1)^2 + 1 = -4 + 4i + 1 + 1 = -2 + 4i$ has reciprocal $(-1-2i)/10$, so $(-2i+4)/10 = 2ai + b$ or $a = -1/10$ and $b = 2/5$. Cover up the $(s+1)^2 + 1$ and set $s = -1 + i$: $\dfrac{-1+i}{(-1+i)^2+4} = ci + d$. $(-1+i)^2 + 4 = 1 - 2i - 1 + 4 = 4 - 2i$ has reciprocal $(2+i)/10$ so $ci + d = (1/10)(-1+i)(2+i) = (1/10)(-3+i)$ or $c = 1/10$ and $d = -3/10$. Thus $X(s) = \dfrac{1}{10}\left(\dfrac{-s+4}{s^2+4} + \dfrac{(s+1)-3}{(s+1)^2+1}\right)$. Thus $x = (1/10)(-\cos(2t) + 2\sin(2t) + e^{-t}(\cos t - 3\sin t))$. I can report that I made more mistakes using this technique than I did using (i).

(iii) We need to use the unit impulse response, computed above as $w(t) = e^{-t}\sin t$ (for $t > 0$).

[1] $f(t) = \delta(t)$: $x(t) = \displaystyle\int_0^t w(t-\tau)\delta(\tau)\,d\tau$. Here we see that we should really have used $0-$ for the lower limit in the convolution integral, so we get $x(t) = w(t)$ from this integral.

[1] $f(t) = u(t)$: $x(t) = \displaystyle\int_0^t w(t-\tau)\,d\tau$ (or $\displaystyle\int_0^t w(\tau)\,dt = \int_0^t e^{-\tau}\sin(\tau)\,d\tau$, using commutativity of the convolution).

[1] $f(t) = \cos(2t)$: $x(t) = \displaystyle\int_0^t w(t-\tau)\cos(2\tau)\,d\tau$.

(b) (i) [2] $f(t)$ is even, so $b_k = 0$ for all k. a_n is $1/\pi$ times the integral over one full "window" of the product $\cos(nt)f(t)$. This window can be taken to be any interval of length 2π. To avoid questions about what happens at the limits of integration, let's take this window to be, say, the interval between $-\pi/2$ and $3\pi/2$. The integral is then $\cos(0) - \cos(n\pi)$, which is 0 if n is even and 2 if n is odd. Thus $f(t) = (2/\pi)(\cos(t) + \cos(3t) + \cos(5t) + \cdots)$.

(ii) [1] The generalized derivative $\mathrm{sq}'(t) = 2f(t)$. Thus

$$\mathrm{sq}(t) = \int \frac{4}{\pi}(\cos(t) + \cos(3t) + \cos(5t) + \cdots)\, dt = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \cdots\right) + c\,.$$

The constant c is the average value of $\mathrm{sq}(t)$, which is zero.

(c) appeared accidentally and is not assigned. Here is an answer anyway. The point is that in the convolution integral $w(t) * q(t) = \int_0^t w(t-\tau)q(\tau)\, d\tau$ the integrand is positive for $0 < \tau < t$, so the integral is positive.

29. **(a)** [3] $\omega_n = \sqrt{25/8} = 5/2\sqrt{2}$. The roots of the characteristic polynomial are $(1/4) \pm (7/4)i$, so the system is underdamped with damped circular frequency $7/4$. Independent real solutions are $e^{-t/4}\cos(7t/4)$ and $e^{-t/4}\sin(7t/4)$.

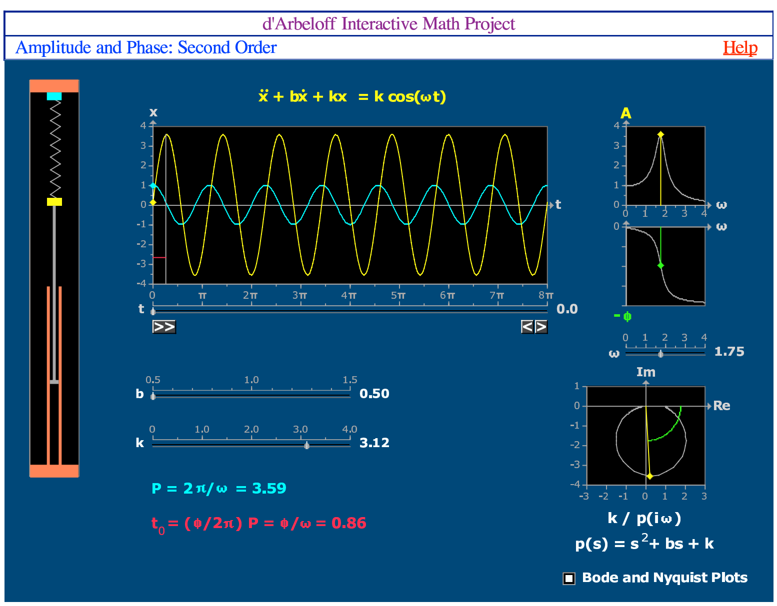
(b) (i) [2] $W(s) = (25/4)/(2s^2 + s + (25/4))$.

(ii) [2] The poles of $W(s)$ occur at the roots of $p(s)$, i.e. at $(1/4) \pm (7/4)i$.

(iii) [2] The graph of $|W(s)|$ is a surface lying over the complex plane, which sweeps up to infinity above the two poles and levels off to zero as you move away from them in any direction. It is always positive, and it is symmetric across the real axis.

(c) [3] (i) $|W(i\omega)| = (25/4)/\sqrt{((25/2) - 2\omega^2)^2 + \omega^2}$.

(ii) Same answer.

(iii) 

(d) [4] The part of the graph of $|W(s)|$ lying over the imaginary axis is exactly the gain graph visualized in **(c)** extended to be an even function of ω.

31. **(a)** [2] $s^2+3s+2=(s+1)(s+2)$ has roots -1 and -2, so $\ddot{x}+3\dot{x}+2x=0$ has independent solutions $x_1=e^{-t}$ and $x_2=e^{-2t}$ (or the other order), general real solution $x=ae^{-t}+be^{-2t}$, and is overdamped. $\dot{x}_1=-e^{-t}$ and $\dot{x}_2=-2e^{-2t}$.

$s^2+2s+4=(s+1)^2+3$ has roots $-1\pm\sqrt{3}i$, so $\ddot{x}+2\dot{x}+4s=0$ has independent real solutions $x_1=e^{-t}\cos(\sqrt{3}t)$ and $x_2=e^{-t}\sin(\sqrt{3}t)$ (or the other order), general real solution $x=e^{-t}(a\cos(\sqrt{3}t)+b\sin(\sqrt{3}t))$ or $x=Ae^{-t}\cos(\sqrt{3}t-\phi)$, and us underdamped. $\dot{x}_1=e^{-t}(-\cos(\sqrt{3}t)-\sqrt{3}\sin(\sqrt{3}t))$, $\dot{x}_2=e^{-t}(\sqrt{3}\cos(\sqrt{3}t)-\sin(\sqrt{3}t))$,

(b) [2] Companion matrices $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}$.

(c) [2] The trajectory of $\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ is the ray eminating from the origin in the direction $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. (It is NOT the entire line, and it does not include the origin.) It is directed inwards. The trajectory of $\begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix}$ is the ray eminating from the origin in the direcion $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. It is also directed inwards.

(d) [2] We have to solve $\ddot{x}+3\dot{x}+2x=0$ with initial conditions $x(0)=0$ and $\dot{x}(0)=1$. Using the general solution from **(a)**, which has $\dot{x}=-ae^{-t}-2be^{-2t}$, we find $a+b=0$ and $-a-2b=1$. Adding, $-b=1$ so $b=-1$ and $a=1$: $x=e^{-t}-e^{-2t}$. Its derivative is $\dot{x}=-e^{-t}+2e^{-2t}$, so the solution is $\begin{bmatrix} e^{-t}-e^{-2t} \\ -e^{-t}+2e^{-2t} \end{bmatrix}$.

(e) [2] The general solution having this trajectory simply shifts time by some constant:

$$\begin{bmatrix} 2e^{-(t-a)}-e^{-2(t-a)} \\ -2e^{-(t-a)}+2e^{-2(t-a)} \end{bmatrix}$$

(f) [2] This solution is a spiral, spiralling clockwise in towards the origin. It pass through the point $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (and it does this at $t=0$, but this information is not part of the phase portrait).

To find the trajectory passing through $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we must solve $\ddot{x}+2\dot{x}+4x=0$ with initial condition $x(0)=0$, $\dot{x}(0)=1$. $x(0)=0$ implies that $a=0$ in the general solution computed in **(a)**, so $\dot{x}(0)=b\dot{x}_2(0)=b\sqrt{3}$ and so $b=1\sqrt{3}$. Thus $x=(1\sqrt{3})e^{-t}\sin(\sqrt{3}t)$ and $\dot{x}=e^{-t}(\cos(\sqrt{3}t)-(1/\sqrt{3})\sin(\sqrt{3}t))$, so a solution with this trajectory is $\begin{bmatrix} (1\sqrt{3})e^{-t}\sin(\sqrt{3}t) \\ e^{-t}(\cos(\sqrt{3}t)-(1/\sqrt{3})\sin(\sqrt{3}t)) \end{bmatrix}$ It crosses the y axis again when $x(t)=0$, or $\sin(\sqrt{3}t)=0$, which happens next at $t=\pi/\sqrt{3}$. The answer is the same for all the solutions with this trajectory.

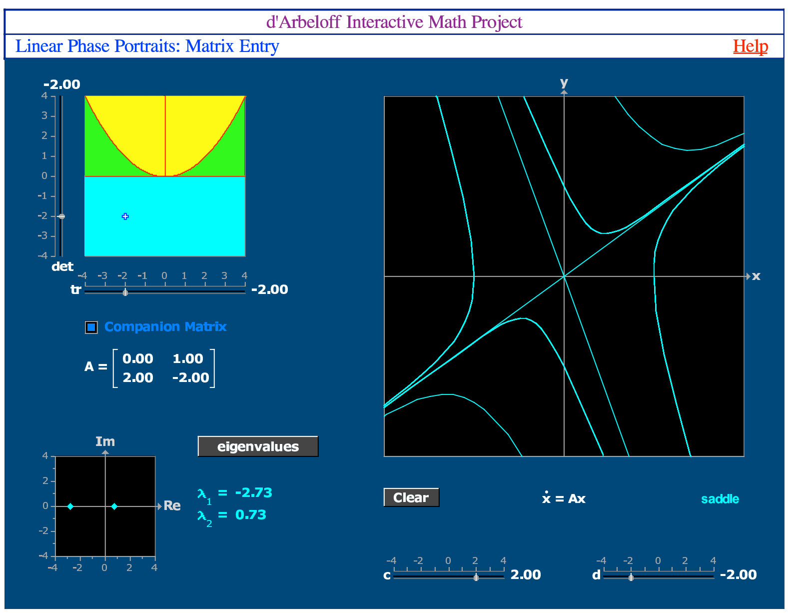
32. **(a)** The characteristic polynomial of of $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ is $\lambda^2+3\lambda+2$, that is, it is the same as the characteristic polynomial of the original second order equation (but with variable called λ instead of s). So the eigenvalues are the roots, -1 and -2. For $\lambda=-1$, we want a nonzero vector $\mathbf{v}_1$ such that $\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\mathbf{v}_1=\mathbf{0}$. $\mathbf{v}_1=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ or any nonzero multiple will serve. For $\lambda = -2$, we want a nonzero vector $\mathbf{v}_2$ such that $\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \mathbf{v}_2 = \mathbf{0}$. $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ or any nonzero multiple will serve.

The eigenlines are the lines through the origin of slope -1 and slope -2. A solution moving along the first is $ce^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (for any nonnzero constant c), and along the second is $ce^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (for any nonzero constant c).

(b) The companion matrix is $\begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix}$. Its characteristic polynomial is the same as the characteristic polynomial of the original equation, $\lambda^2 + 2\lambda - 2$. Its roots are $-1 \pm \sqrt{3}$.

An eigenvector $\mathbf{v}_1$ for $-1 + \sqrt{3}$ satisfies $\begin{bmatrix} 1 - \sqrt{3} & 1 \\ 2 & -1 - \sqrt{3} \end{bmatrix} \mathbf{v}_1 = \mathbf{0}$. $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 - \sqrt{3} \end{bmatrix}$ or any nonzero multiple will do.

An eigenvector $\mathbf{v}_2$ for $-1 - \sqrt{3}$ satisfies $\begin{bmatrix} 1 + \sqrt{3} & 1 \\ 2 & -1 + \sqrt{3} \end{bmatrix} \mathbf{v}_2 = \mathbf{0}$. $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 + \sqrt{3} \end{bmatrix}$ or any nonzero multiple will do.



The rays with positive slope flee from the origin, and the rays with negative slope converge to the origin.

Normal modes are given by $ce^{(-1+\sqrt{3})t} \begin{bmatrix} -1 \\ 1 - \sqrt{3} \end{bmatrix}$ and $ce^{(-1-\sqrt{3})t} \begin{bmatrix} -1 \\ 1 + \sqrt{3} \end{bmatrix}$.