18.03 Problem Set 9

Due by 1:00 P.M., Friday.

V. First order systems

R20	Th 27 Apr	Matrices and column vectors
L31	F 28 Apr	Linear systems and matrices: EP 5.1–5.3, SN 24, Notes LS.1.
L32	M 1 May	Eigenvalues, eigenvectors: EP 5.4, Notes LS.2.
R21	T 2 May	ditto
L33	W 3 May	Complex or repeated eigenvalues: EP 5.4, Notes LS.3.
R22	Th 4 May	ditto
L34	F 5 May	Qualitative behavior of linear systems: SN 25.
L35	M 8 May	Normal modes and the matrix exponential: EP 5.7, Notes LS.6.
R23	T 9 May	ditto
L36	W 10 May	Inhomogeneous equations: variation of parameters again: EP 5.8.
R24	Th 11 May	ditto
L37	F 12 May	Nonlinear systems: EP 7.2, 7.3; Notes GS.
L38	M 15 May	Examples of nonlinear systems: EP 7.4, 7.5; Notes GS.
R25	T 16 May	ditto
L39	W 17 May	Chaos.

Part I.

33. (W 3 May) Notes: 4D-2. Recitation 23 problem: find basic real solutions of $\dot{\mathbf{u}} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{u}$.

34. (F 5 May) None.

35. (M 8 May) Notes 4G-1(a), and find e^{tA}. Also, find A itself. 4G-2.

36. (W 10 May) Notes 4I-1.

Part II.

33. (W 3 May) [Complex or repeated eigenvalues]

(a) Find two basic solutions for $\dot{\mathbf{u}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{u}$.

(b) $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ is a companion matrix. What is the corresponding second order equation? Find two linearly independent solutions, say x_1 and x_2, for it. Write $\begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ \dot{x}_2 \end{bmatrix}$ as linear combinations of the basic solutions you found in **(a)**.

34. (F 5 May) [Qualitative behavior of linear systems] Invoke the Mathlet `Linear Phase Portraits: Matrix Entry`. Play with the tool for a while to get a feel of it. Notice that the eigenvalues can be displayed on the complex plane. Deselect the `[Companion Matrix` option, so you can set all four entries in the matrix. Select the `[eigenvalues]` option, so the eigenvalues become visible by means of a plot of their location in the complex plane and also a read-out of their values.

We will use this tool to investigate the phase portraits of the homogeneous linear equation $\dot{\mathbf{u}} = A\mathbf{u}$, where $A = \begin{bmatrix} 1 & 3 \\ -1 & d \end{bmatrix}$, as d varies. To start with, set the matrix to $A = \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}$. Then move the d slider up to $d = 4$, and watch (1) the movement of the mark on the (Tr,Det) plane; (2) the movement of the eigenvalues in the complex plane; and (3) the variation of the phase plane.

(a) Compute the trace and determinant of A. (They will depend upon a, of course.) Find an equation for the curve (or line) traced out by the mark on the (Tr,Det) plane.

(b) You notice that the curve in the (Tr,Det) plane enters a number of different regions. When the cursor crosses a red boundary, the trajectories and the eigenvalue indicators turn red. Work out what the values of d are at those crossings. (So this is: where $\det A = 0$, where $\det A = (\mathrm{tr}A/2)^2$ (twice, once not represented on the Mathlet), and where $\mathrm{tr}A = 0$.

(c) There are nine phase portrait types represented as d varies (five regions and four walls). Draw an interval from -4 to $+5$. On it, mark the four values of d at which the matrix crosses one of the walls. Indicate the type of phase portrait you have at each of the marked points and along the intervals between them. That is, classify the phase portrait into one of the following types, as in the Supplementary Notes, §25: spiral (stable/unstable, clockwise/counterclockwise), node (stable/unstable); saddle; center (clockwise/counterclockwise); star (stable/unstable); defective node (stable/unstable; clockwise/counterclockwise); degenerate (comb (stable/unstable), constant, parallel lines).

(d) For each of the four special values, and for your choice of one value in each of the five regons, make a sketch of the phase portrait. Be sure to include and mark as such any eigenlines, and the direction of time.

35. (M 8 May) [Matrix exponential] Find e^{At} where

(a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$. (Hint: use the work from I.33.)

(b) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. **(c)** $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

36. (W 10 May) [Inhomogeneous equations]

Solve the equation $\dot{\mathbf{u}} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{u} + \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ with initial condition $\mathbf{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.