Recitation 1, February 7, 2006

Natural growth and decay

Solution suggestions

1. Write down a model for the ant population.

Let us assume that at t days we have $N(t)$ ants in the square. We have to determine how this number changes over a time of Δt days. For example, Δt could be 1/2, and we would look at the change of $N(t)$ over the next 12 hours starting at t days. First, there is the reproduction of the ants. Within the time Δt we have $k\,\Delta t\,N(t)$ new ants as k is the number of new ants *per ant* per day. It would look like that we have a number of $N(t) + k\ N(t)\,\Delta t$ ants after Δt days. But we haven't taken into account the emigration of some ants. Within the time of Δt days a number of $a\ \Delta t$ has emigrated. Remember that a was the number of ants that emigrates over a *whole* day. Thus, we get to

$$N(t + \Delta t) \simeq N(t) + k\,N(t)\,\Delta t - a\,\Delta t.$$

Rearranging this equation in the same way as we did for the radioactive decay we obtain

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} \simeq k\,N(t) - a. \tag{1}$$

Now, we have to ask ourselves the question what if we had first computed the number of emigrated ants and *then* thought about the reproduction. After the emigration, we would have had only $N(t) - a\,\Delta t$ ants left. Then, we would get

$$N(t + \Delta t) \simeq N(t) - a\,\Delta t + k\Big(N(t) - a\,\Delta t\Big)\,\Delta t.$$

Rearranging this equation in the same way as before we obtain

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} \simeq k\,N(t) - a - k\,a\,\Delta t. \tag{2}$$

Comparing our two approaches we see that in the first case we assumed that *all* of the reproduction occured at t days, and in the second case we assumed that all of the reproduction occured at $t+\Delta t$ days. Both are only two simplifications of what really happens. However, in the end Δt will be small. In fact, we want to obtain the derivative on the LHS of Eqs. (1) and (2). If we make Δt small the difference of Eqs. (1) and (2) is small compared to $k\,N(t) - a$. Therefore, we will neglect the term $k\,a\,\Delta t$ on the RHS of Eq. (2). Then, it doesn't matter which approach we choose. We obtain from Eqs. (1) and (2) the differential equation

$$\dot{N}(t) = k\,N(t) - a. \tag{3}$$

2. Find the general solution of this equation.

We can cross multiply Eq. (3) to obtain

$$\frac{dN}{N - \frac{a}{k}} = k. \tag{4}$$

By integration, this becomes

$$\ln\left|N - \frac{a}{k}\right| = k\,t + C_1\ .$$

Solving for N, we finally get

$$N = N(t) = \frac{a}{k} \pm e^{C_1} e^{k\,t}\ .$$

It is better to write

$$N = N(t) = \frac{a}{k} + C\,e^{k\,t}$$

where C is now an arbitrary real constant. The value zero for C is now included so that we don't miss any solution. As in the case of the radioactive decay, we like to express C through the number N_0 of ants we had when we started the experiment. We write

$$N_0 = N(t=0) = \frac{a}{k} + C.$$

Thus $C = N_0 - a/k$ and the final answer is

$$N(t) = \frac{a}{k} + \left(N_0 - \frac{a}{k}\right) e^{kt}. \tag{5}$$

3. Check that the proposed solution satisfies the ODE.

We take the derivative of our proposed solution (5) and obtain

$$\dot{N}(t) = 0 + k\left(N_0 - \frac{a}{k}\right) e^{kt}.$$

We also check

$$k\,N(t) - a = a + k\left(N_0 - \frac{a}{k}\right) e^{kt} - a = k\left(N_0 - \frac{a}{k}\right) e^{kt}.$$

Thus, $N(t)$ is a solution to the differential equation (3).

4. There is a "steady state" (also known as constant, or equilibrium) solution. Find it. Does the way the solution depends upon k and a make sense? (That is: do the units come out right? Does it behave right when a is large, or small, and when k is large, or small?)

We see that the steady state solution is $N(t) = a/k$. If we set our starting population N_0 equal to a/k ants then it remains at this number for all times. It means that we have an equilibrium between some ants leaving and them being replaced by reproduction. The units for k are $1/[day]$ and for a they are $[ants]/[day]$, so the ratio a/k gives in fact a number of ants.