Recitation 3, February 14, 2006

Numerical methods; Linear models

Solution suggestions

1. Use Euler's method to estimate the value at $x = 1.5$ of the solution of $y' = y^2 - x^2 = F(x, y)$ at with $y(0) = -1$. Use $h = 0.5$. Make the table: k, x_k,y_k, $A_k = F(x_k, y_k)$, $h \cdot A_k$. Draw the Euler polygon.

k	x_k	y_k	A_k	$h\,A_k$
0	0.0	-1.000	1.000	0.500
1	0.5	-0.500	0.000	0.000
2	1.0	-0.500	-0.750	-0.375
3	1.5	-0.875		

2. Is the estimate from **1.** too large or to small?

If the direction field is becoming steeper under the segment of the Euler polygon, then the Euler estimate is low; if it's becoming less steep, then the Euler estimate is too high. Since $A_1 < A_0$, it looks like the estimate is too high.

3. A tank holds V liters of salt water. Suppose that a saline solution with concentration of c gm/liter is added at the rate of r liters/minute. A mixer keeps the salt essentially uniformly distributed in the tank. A pipe lets solution out of the tank at the same rate of r liters/minute. Write down the differential equation for the *amount* of salt in the tank. [Not the concentration!] Check the units in your equation! Write it in standard form.

We have to determine how the amount of $x(t)$ gm of salt changes over a small time interval Δt minutes. First, we determine the amount of salt that we are adding over Δt minutes. The saline solution that is flowing in adds the amount of $c\,r\,\Delta t$ gm of salt. The units of c, r, and Δt combine to give gm. Second, we have to determine how much salt is flowing out. Remember that we are assuming that at all times the solution in the tank is kept uniformly mixed. That means that at time t minutes the concentration of salt in the tank is $x(t)/V$ gm/liter. The *amount* flowing out is thus $\frac{x(t)}{V}\,r\,\Delta t$. Again, the units combine to gm. Combining these contributions we obtain

$$\Delta x \simeq c\,r\,\Delta t - \frac{x}{V}\,r\,\Delta t\ .$$

Every term in this equation describes an amount of salt in gm. Dividing by Δt and taking the limit we obtain the differential equation

$$\dot{x} = c\,r - \frac{x}{V}\,r\ .$$

Written in its standard form this is

$$\dot{x} + \frac{r}{V}\,x = c\,r\ .$$

4. Now assume that c and r are constant; in fact, assume that $V = 1$ and $r = 2$. Solve this equation, under the assumption that $x(0) = 0$

What is the limiting amount of salt in the tank? Does your result jibe with simple logic? When will the tank contain half that amount?

We can solve this differential equation by separation of variables. Setting $V = 1$ and $r = 2$ and cross multiplying we obtain

$$\frac{dx}{c-x} = 2dt$$

By integration we obtain

$$-\ln|c-m| = 2t + C_1 .$$

Exponentiating, eliminating the absolut values, and returning the 'missing solution' we find

$$c - x = C_2\, e^{-2t} .$$

At $t = 0$ we have $x(0) = 0$, so $C_2 = c$. Thus, we obtain

$$x = c\Big(1 - e^{-2t}\Big) .$$

For large t the term e^{-2t} goes to zero, and the limiting amount is $x = c$ gm. This agrees with intuition. We start with no salt in the tank, then poor in saline solution of a certain concentration, and *after mixing* remove some of it. After a very long time we should have a an equilibrium of what is going in and what is flowing out. Therefore, the concentration in the tank must be the concentration of the inflowing saline solution.

Now, we determine at what time T we have half the limiting amount. This means we need to determine T such that $x(T) = c/2$ or

$$c\Big(1 - e^{-2T}\Big) = c/2 .$$

Solving for T we obtain $T = \ln(2)/2$ minutes, approximately 21 seconds.

5. Now suppose that the out-flow from this tank leads into another tank, also of volume 1, and that at time $t = 0$ the water in it has no salt in it. Again there is a mixer and an outflow. Write down a differential equation for the amount of salt in this second tank, as a function of time.

We set up an equation describing the change of the amount of $y(t)$ gm of salt in the second tank. Following the reasoning from problem 4 we obtain

$$\Delta y \simeq= \frac{x}{1}\, r\, \Delta t - \frac{y}{1}\, r\, \Delta t .$$

Remember that the outflow of the first tank has concentration $x(t)/1$ gm/liter and that the volume of the second tank is 1 liter as well. Thus we obtain the differential equation ($r = 2$)

$$\dot{y} = 2c\left(1 - e^{-2t}\right) - 2\,y .$$

The inital condition is $y(0) = 0$. The initial condition is stating that at $t = 0$ (*not at* $t = 1$ *as originally stated*) there is no salt in tank 2.