Recitation 6, February 28, 2006

Using the complex exponential; Autonomous equations

Solution suggestions

1. Find the sinusoidal solution of $\dot{x}+2x=\cos(2t)$ in polar form, $A\cos(\omega t-\phi)$, in the following way: First find the exponential solution of the corresponding equation with complex exponential right hand term; it is $z_p=\dfrac{1}{2i+2}e^{2it}$. Then find A and ϕ such that $\dfrac{1}{2i+2}=Ae^{-i\phi}$, and use this to re-express $z_p=Ae^{(2t-\phi)i}$. Now take the real part.

The corresponding complex differential equation is $\dot{z}+2z=e^{2it}$. The desired solution x is going to be the real part of a solution z. Writing $z_p=Ae^{2it}$ and plugging it into the ODE we obtain

$$(2i+2)Ae^{2it}=e^{2it}\ ,$$

thus $A=\frac{1}{2+2i}$ or $z_p=\dfrac{1}{2i+2}e^{2it}$. Now, we find A and ϕ such that $\dfrac{1}{2i+2}=Ae^{-i\phi}$. We write

$$\frac{1}{2+2i}=\frac{2-2i}{8}=Ae^{-i\phi}\ .$$

To determine A and
phi we draw a right triangle with sides $a=\frac{1}{4}$ and $b=\frac{1}{4}$. Remember that by the Euler-formula we have

$$e^{-i\phi}=\cos\phi-i\sin\phi\ .$$

This is why we haven taken both a and b positive. Now, the length of the hypotenuse is $A=\frac{\sqrt{2}}{4}$. The angle between the side a and the hypotenuse is $\phi=\frac{\pi}{4}$ (45 degrees). We obtain

$$z_p=\frac{1}{2+2i}e^{2it}=Ae^{-i\phi}\,e^{2it}=Ae^{i(2t-\phi)}\ ,$$

with $A=\frac{\sqrt{2}}{4}$ and $\phi=\frac{\pi}{4}$. In summary, we have

$$z_p=\frac{\sqrt{2}}{4}e^{i(2t-\frac{\pi}{4})}\ .$$

Now we take the real part and obtain

$$x_p=\frac{\sqrt{2}}{4}\cos(2t-\frac{\pi}{4})\ .$$

2. The growth-rate of the population of Ivory-billed Woodpeckers falls to zero as the population density does, because there are just so few of them that it's hard to find a mate. On the other hand, they require many acres of forest and compete with other IBWs when the population grows. This can be modeled using a growth-rate of $k(y) = 4k_0 \frac{y}{p}\left(1 - \frac{y}{p}\right)$, where k_0 and p are constants.

Show that the maximal growth-rate is k_0; this explains the factor $4k_0$ in the front.

What is the autonomous equation for $\dot{y}$? (NB: it's not $\dot{y} = k(y)$.)

Sketch the phase line and some solutions. Classify the critical points: stable, unstable, semi-stable.

Pressure from bird watchers reduces the population growth by a constant rate a. How high can that rate go before the IBWs face certain extinction? (First find the IBW population where that maximum "harvest rate" occurs.)

First, let us look at the growth-rate: we observe that $k(y)$ is zero for $y = 0$ and $\frac{y}{p} = 1$. The function $k(y)$ is also positive for $0 < y < p$ and negative for $y > p$. Now, we find the critical points of $k(y)$. We set

$$0 = k'(y) = 4k_0 \frac{1}{p}\left(1 - \frac{y}{p}\right) - 4k_0 \frac{y}{p^2} = \frac{4k_0}{p}\left(1 - \frac{2y}{p}\right) .$$

We find the solutions $y = 0$ and $y = \frac{p}{2}$. Therefore, the growth-rate has a maximum at $y = \frac{p}{2}$ and this maximum value is k_0.

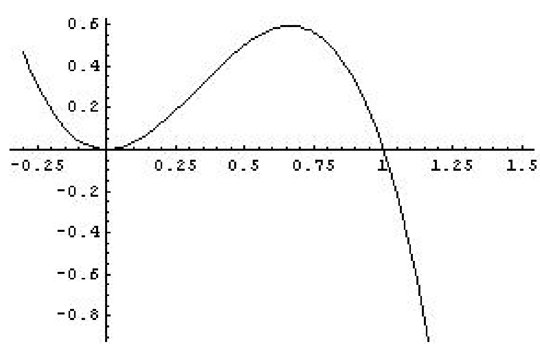
Now, let us determine the autonomous ODE describing the population of IBWs: Since $k(y)$ is the growth rate it means for the change of the population $y(t)$ over Δt that

$$y(t + \Delta t) \simeq y(t) + k(y)\, y(t)\, \Delta t .$$

Remember that growth-rate means that for every IBW extant at time t, about $k(t)\Delta t$ more IBWs appear over the time interval Δt. In conclusion, we have the autonomous ODE

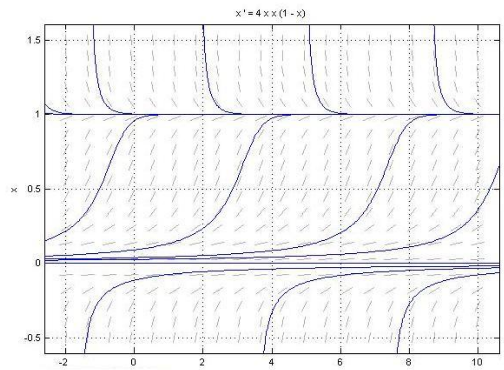
$$\dot{y} = k(y) y .$$

In the notation of the lecture we would write $g(y) = k(y)\, y$. Here is a graph of the function $g(y)$:



Looking at the graph of $g(y)$ we observe that $y = p$ is a stable critical point, the point $y = 0$ is semistable. Direction arrows point up below 0, up between 0 and p and down above p.

Now, we can sketch the phase line.



The birdwatchers have the effect of subtracting a from the formula for $\dot{y}$. This has the effect of pushing the graph of $g(y)$ down by a units.

$$\dot{y} = k(y)\, y - a \ .$$

Notice that there's a difference between 'growth' and 'growth rate.' Now, how big can a be before we face extinction? When the maximum value y_{max} for $y > 0$ hits the y axis, we are in trouble. So we have to find the maximum value of $g(y)$, by differentiating it and setting it equal to zero. Therefore, we have to solve

$$0 = \frac{d}{dy}\Big|_{y=y_{max}} g(y) = k'(y_{max})y_{max} + k(y_{max})$$

for y_{max}. The RHS is

$$k'(y_{max})y_{max} + k(y_{max}) = \frac{4k_0 y_{max}}{p}\left(2 - \frac{3y_{max}}{p}\right) \ .$$

Thus, $y_{max} = \frac{2}{3}p$. For the population where we have the maximum 'harvest rate' we have

$$g(y_{max}) - a = 0 \ .$$

Thus, $a = y_{max}k(y_{max}) = \frac{2}{3}p\ k(\frac{2}{3}p) = \frac{16k_0p}{27}$.