Recitation 16, April 11, 2006

Step and delta functions, and step and delta responses

Solutions suggestions

1. Graph the functions

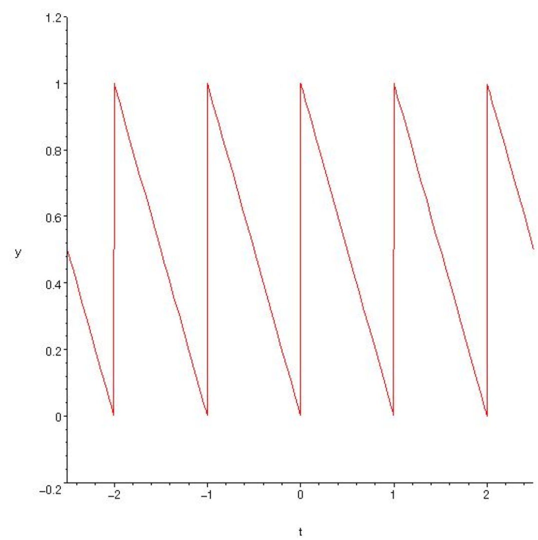
$$f(t) = 1 + \lfloor t \rfloor - t$$

(where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t) and

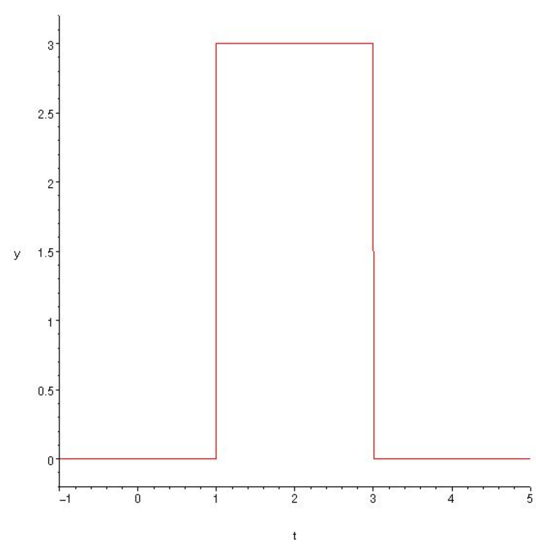
$$g(t) = 3(u(t-a) - u(t-b))$$

(where $a < b$). Then find their generalized derivatives and graph them, using harpoons to denote the delta functions that occur.

Ans. Here is the graph for the function $f(t)$:



Here is the graph of the function $g(t)$ (for $a = 1$ and $b = 3$):



To determine the generalized derivative of $f(t)$, we first notice that $f(t)$ is periodic with period 1. Therefore, it is enough to look at t in the range $-1/2 \le t \le 1/2$. Since $f(-1/2) = f(1/2)$ the function is continuous at $1/2$, and we see that the only kink in the graph of $f(t)$ between $-1/2 \le t \le 1/2$ appears at $t = 0$. Let's write down the possible values for $f(t)$:

$$f(t) = \begin{cases} -t & -\frac{1}{2} < t < 0 \\ 1-t & 0 < t \le \frac{1}{2} \end{cases} ,$$

or for t with $-1/2 \le t \le 1/2$ we can write $f(t) = -t + u(t)$. The generalized derivative is

$$\dot{f}(t)\Big|_{-\frac{1}{2}\le t\le\frac{1}{2}} = -1 + \delta(t) .$$

The function $f(t)$ is periodic with period 1. This means that the graph of $f(t)$ for t with $1/2 < t < 3/2$ looks exactly like the graph of $f(t)$ for t with $-1/2 < t < 1/2$ (i.e. shifted by one unit to the right). This means

$$f(t) = \begin{cases} -(t-1) & \frac{1}{2} < t < 1 \\ 1-(t-1) & 1 < t \le \frac{3}{2} \end{cases}$$

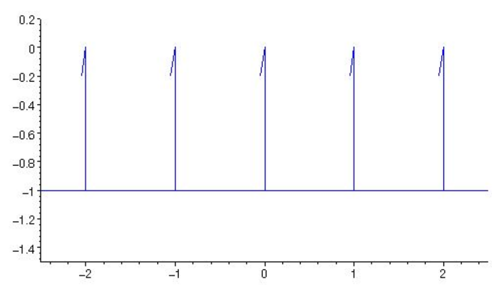
where we have just replaced t by $t-1$. Accordingly, the derivative is

$$\dot{f}(t)\Big|_{\frac{1}{2}\le t\le\frac{3}{2}} = -1 + \delta(t-1) .$$

Repeating this argument, we find

$$\dot{f}(t) = -1 + \sum_{n=-\infty}^{\infty} \delta(t-n) .$$

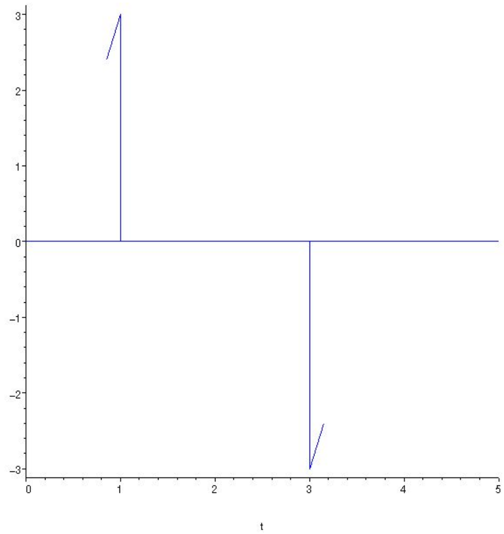
Here is the graph of $\dot{f}(t)$ (with harpoons denoting the occuring delta functions):



Similarly, we compute for $g(t)$

$$\dot{g}(t) = 3\delta(t-a) - 3\delta(t-b).$$

Here is the graph of the function $\dot{g}(t)$ (for $a = 1$ and $b = 3$):



2. Find the unit step and unit impulse responses to the operator $mD^2 - kI$, for $m > 0$, and graph them.

Ans. Let's start with computing the unit step response: the differential equation we seek to solve is

$$(mD^2 - kI)x(t) = u(t) \ .$$

We can write this as

$$m\ddot{x}(t) - kx(t) = u(t) \ . \tag{1}$$

For $t > 0$, Eq. (1) is the same as

$$m\ddot{x}(t) - kx(t) = 1$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. A particular solution is given by $x_p(t) = -1/k$. To obtain the general solution we have to add the homogenous solution, i.e. the general solution of

$$m\ddot{x}(t) - kx(t) = 0$$

which is $Ae^{w\,t} + Be^{-w\,t}$ where $w^2 = k/m$. From recitation 15 we know that we can also write this solution as an even plus an odd part, i.e. $a\cosh(wt) + b\sinh(wt)$. Together, the solution becomes

$$x(t) = -\frac{1}{k} + a\cosh(w\,t) + b\sinh(w\,t) \ .$$

We want $x(0) = 0$ and this gives $a = 1/k$. Thus, we obtain

$$x(t) = -\frac{1}{k} + \frac{1}{k}\cosh(w\,t) + b\sinh(w\,t) \ .$$

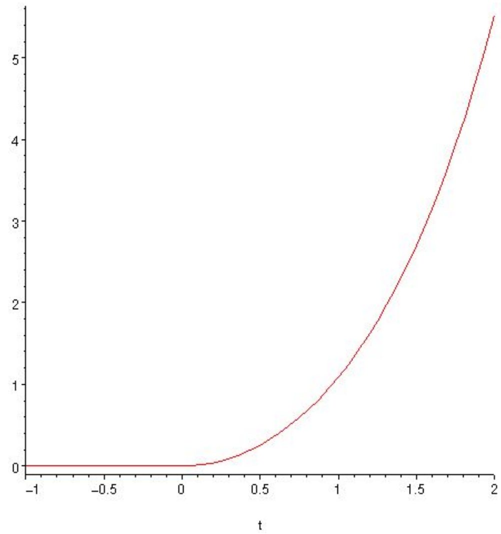
We compute $\dot{x}(t)$:

$$\dot{x}(t) = \frac{w}{k}\sinh(wt) + bw\,\cosh(wt) \ .$$

As we also want $\dot{x}(0) = 0$ we find $b = 0$. In summary, the unit step response is given by

$$x(t) = \begin{cases} -\frac{1}{k} + \frac{1}{k}\cosh(wt) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} .$$

Here is the graph of $x(t)$ for $k = 1/2$ and $m = 1/2$:



Let's compute the unit impulse response: the differential equation we seek to solve is

$$(mD^2 - kI)y(t) = \delta(t) .$$

We can write this as

$$m\ddot{y}(t) - ky(t) = \delta(t) . \qquad (2)$$

For $t > 0$ Eq. (2) is the same as

$$m\ddot{y}(t) - ky(t) = 0$$

with initial conditions $y(0) = 0$ and $m\dot{y}(0) = 1$. Again, the general solution of

$$m\ddot{y}(t) - ky(t) = 0$$

is $a\cosh(wt) + b\sinh(wt)$ with $w^2 = k/m$. Thus

$$y(t) = a\cosh(wt) + b\sinh(wt) .$$

We want $y(0) = 0$ thus $a = 0$, and

$$y(t) = b\sinh(wt)$$

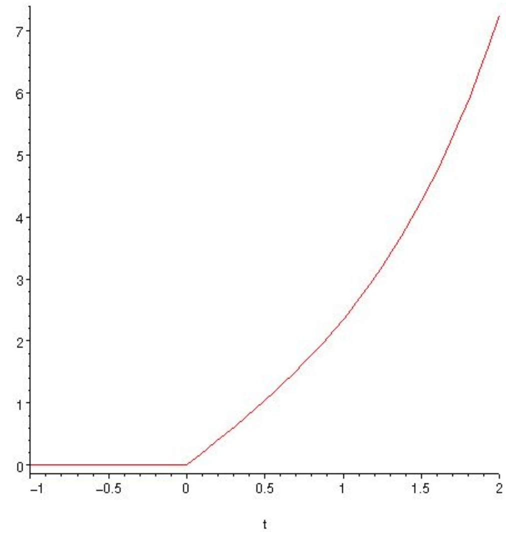
and

$$\dot{y}(t) = bw\cosh(wt) .$$

As we also want $m\dot{y}(0) = 1$ we find $b = 1/(mw)$. In summary, the unit step response is given by

$$y(t) = \begin{cases} \frac{1}{mw}\sinh(wt) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} .$$

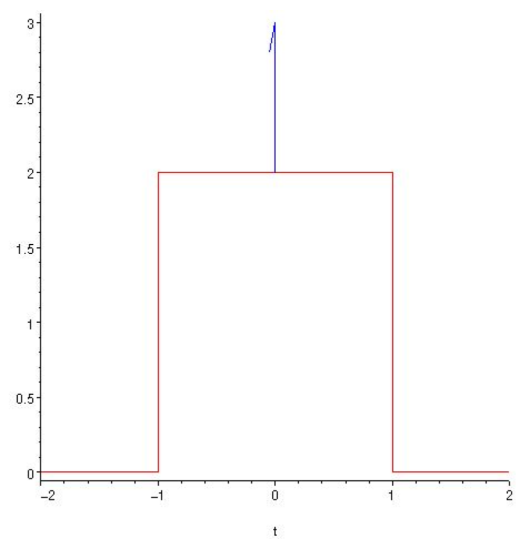
Here is the graph of $y(t)$ for $k = 1/2$ and $m = 1/2$:



Since $w = \sqrt{\frac{k}{m}}$, we have $\frac{w}{k} = \frac{1}{mw} = \frac{1}{\sqrt{km}}$. Then, we see that $y(t) = \dot{x}(t)$.

3. Suppose $q(t) = 2u(t+1)+\delta(t)-2u(t-1)$. Sketch a graph of this generalized function. Tell stories which might result in each of the equations $\dot{x}+kx = q(t)$ (your choice of k, it might be negative) and $2\ddot{x} + 4\dot{x} + 18x = q(t)$.

Ans. Here is the graph of $q(t)$:



A driven spring/mass/dashpot system is described by the ODE

$$m\ddot{x} + b\dot{x} + kx = F_{ext}(t)$$

where we set $m = 2$, $b = 4$, $k = 18$. Now, we want to design the external force F_{ext} to fit $q(t)$. If we consider a hammer blow large enough to increase the momentum $m\dot{x}(0)$ by one unit, then this would give us an external force of just $\delta(t)$. If we applied a constant force of magnitude 2 in the positive x-direction but only between $-1 < t < 1$ this would result in an external force of $2[u(t+1)-u(t-1)]$. Combining the constant force and the hammer blow gives the right externernal force. The system is then described by the differential equation

$$2\ddot{x} + 4\dot{x} + 18x = \delta(t) + 2[u(t+1) - u(t-1)] \; .$$

Now, we want to describe a situation which is modelled by the first order differential equation. Let us look at a nuclear reactor: the amount of plutonium $x(t)$ present in the reactor's core is described by the differential equation

$$\dot{x} + kx = M(t)$$

where $M(t)$ describes how much plutonium we place in the reactor core. If we just put a single amount of plutonium in at $t = 0$ this would correspond to setting $M(t)$ equal to $\delta(t)$. If we started loading the reactor at the constant rate 2, but only between $-1 < t < 1$ this would correspond to setting $M(t)$ equal to $2[u(t+1) - u(t-1)]$. Combining the constant loading and the single placement of plutonium in the reactor core gives the differential equation

$$\dot{x} + kx = \delta(t) + 2[u(t+1) - u(t-1)] \; .$$

4. Find the unit step and unit impulse responses for $2D^2 + 4D + 20I$. Why is one the derivative of the other?

Ans. We are looking for the solutions of the differential equations

$$\text{unit step response w/ rest initial conditions:} \qquad p(D)x(t) = u(t) \tag{3}$$

and

$$\text{unit impulse response w/ rest initial conditions:} \qquad p(D)y(t) = \delta(t) \tag{4}$$

where

$$p(D) = 2D^2 + 4D + 20I$$

and rest initial conditions means

$$x(t) = 0 \qquad \text{for } t < 0$$

and the same for $y(t)$.

For $t > 0$ the unit step response (3) is the same as

$$2\ddot{x} + 4\dot{x} + 20x = 1$$

with initial conditions

$$x(0) = 0 \; , \qquad \dot{x}(0) = 0 \; .$$

As $p(s)$ has roots $-1 \pm i3$, the general solution to $p(D)x = 1$ is

$$x = \frac{1}{20} + e^{-t}\Big(a\cos(3t) + b\sin(3t)\Big).$$

We want $x(0) = 0$, thus $a = -1/20$ and

$$x = \frac{1}{20} + e^{-t}\Big(-\frac{1}{20}\cos(3t) + b\sin(3t)\Big)$$

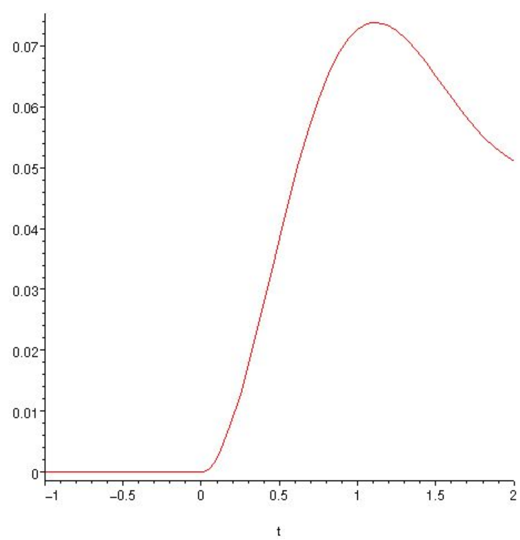
and therefore

$$\dot{x} = e^{-t}\Big[\Big(3b + \frac{1}{20}\Big)\cos(3t) - \Big(\frac{3}{20} + b\Big)\sin(3t)\Big] .$$

Therefore, we have $\dot{x}(0) = 3b + \frac{1}{20}$. We want $\dot{x}(0) = 0$, thus $b = -\frac{1}{60}$. In summary, we have found

$$x(t) = \begin{cases} \frac{1}{20} - \frac{1}{20}e^{-t}\Big(\cos(3t) + \frac{1}{3}\sin(3t)\Big) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} .$$

Here is the graph of $x(t)$:



For $t > 0$ the unit impulse response (4) is the same as

$$2\ddot{y} + 4\dot{y} + 20y = 0$$

with initial conditions

$$y(0) = 0 , \qquad 2\dot{y}(0) = 1 .$$

As $p(s)$ has roots $-1 \pm i32$, the general solution to $p(D)y = 0$ is

$$y = e^{-t}\Big(a\cos(3t) + b\sin(3t)\Big).$$

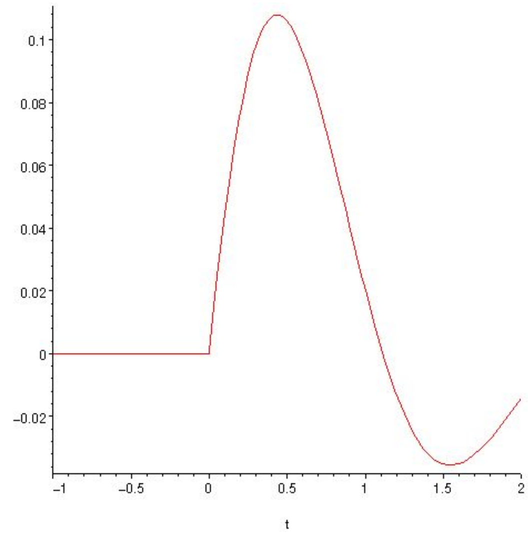
We want $y(0) = 0$, so $a = 0$ and $y = be^{-t}\sin(3t)$. We compute

$$\dot{y} = be^{-t}\Big(3\cos(3t) - \sin(3t)\Big) .$$

Therefore, we have $\dot{y}(0) = 3b$. We want $2\dot{y}(0) = 1$. Thus, $3b = 1/2$ or $b = \frac{1}{6}$. In summary, we have found

$$y(t) = \begin{cases} \frac{1}{6}e^{-t}\sin(3t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} .$$

Here is the graph of $y(t)$:



We can check that $y(t) = \dot{x}(t)$. There is good reason for this: let us assume that we have any solution $x(t)$ for Eq. (3) then it must satisfy

$$p(D)x(t) \quad = \quad u(t) \; .$$

Let's take the derivative $D = \frac{d}{dt}$ on both sides, i.e.

$$D\,p(D)x(t) \quad = \quad Du(t) \; .$$

We know that $Du(t) = \delta(t)$. We check that

$$D\,p(D) = D\Big(2D^2 + 4D + 20I\Big) = 2D^3 + 4D^2 + 20D = p(D)\,D \; .$$

This is nothig but saying that $p(D)\,D = D\,p(D)$ because the coefficients are constant. Therefore, $y(t) = \dot{x}(t) = Dx(t)$ must be a solution to

$$p(D)y(t) \quad = \quad p(D)\,Dx(t) = D\left[p(D)x(t)\right] = Du(t) = \delta(t) \; .$$

We just showed that if $x(t)$ is a solution to the unit step response (3) then $y(t) = \dot{x}(t)$ is a solution to the unit impulse response (4).