18.03 Recitation 18, April 20, 2006

Laplace transform

Solution suggestions

1. Use the s-shift rule and the formulas for $\mathcal{L}[\cos(\omega t)]$ and $\mathcal{L}[\sin(\omega t)]$ to find $\mathcal{L}[e^{at}\cos(\omega t)]$ and $\mathcal{L}[e^{at}\sin(\omega t)]$.

Ans. We obtain

$$\mathcal{L}[e^{at}\cos(\omega t)] = \frac{s-a}{(s-a)^2+\omega^2} \ ,$$

and

$$\mathcal{L}[e^{at}\sin(\omega t)] = \frac{\omega}{(s-a)^2+\omega^2} \ .$$

2. Find the unit impulse and unit step responses of the operator $2D + 4I$ using Laplace transform methods. What is the Laplace transform of the unit impulse and unit step response of the operator $aD + bI$ (for $a \neq 0$)?

Ans. We are looking for the unit impulse and unit response solutions of the operator $p(D) = aD + b$. Thus, we have to solve the following differential equations:

$$\text{unit impulse response w/ rest initial conditions:} \qquad a\,\dot{w}(t) + b\,w(t) = \delta(t) \tag{1}$$

and

$$\text{unit step response w/ rest initial conditions:} \qquad a\,\dot{v}(t) + b\,v(t) = u(t) \ . \tag{2}$$

Let's look at the unit impulse response (1): The weight function of $p(D)$ is the solution to $p(D)w = \delta(t)$ with rest initial conditions. We apply Laplace transform to obtain

$$W(s) = \mathcal{L}[w(t)] = \frac{1}{p(s)} = \frac{1}{as+b} = \frac{1}{a}\frac{1}{s+\frac{b}{a}} \ .$$

The function $w(t)$ whose Laplace transform is $W(s)$, then is

$$w(t) = \mathcal{L}^{-1}[W(s)] = \frac{1}{a}\mathcal{L}^{-1}\left[\frac{1}{s+\frac{b}{a}}\right] = \frac{1}{a}e^{-\frac{b}{a}t} \ ,$$

for $t > 0$. For $t < 0$ we have $w(t) = 0$. We check that we have a discontinuity at $t = 0$, i.e. $w(0-) = 0$ and $aw(0+) = 1$.

Now let's look at the unit step response (2): The unit step response $v(t)$ is the solution to $p(D)v = u(t)$ with rest initial conditions. We apply Laplace transform to obtain

$$p(D)V(s) = 1/s \ .$$

or $V(s) = \frac{1}{s\,p(s)}$ since $p(s) = as + b$. To obtain the function whose Laplace transform is $V(s)$ we have to do a partial fraction decomposition. This means that we want to write

$$\frac{1}{sp(s)} = \frac{1}{s(as+b)} = \frac{m}{s} + \frac{n}{as+b} \,,$$

and then determine m and n. To obtain m, we multiply through with s and then plug in $s = 0$. We obtain $1/b = m$. To obtain n, we multiply through with $as + b$ and then plug in $s = -\frac{b}{a}$ since for $s = -\frac{b}{a}$ the term $as + b$ is zero. Then, we obtain $-a/b = n$. Therefore, we can write $V(s)$ as

$$V(s) = \frac{1}{s\,p(s)} = \frac{1}{b}\left(\frac{1}{s} - \frac{a}{as+b}\right) = \frac{1}{b}\left(\frac{1}{s} - \frac{1}{s+\frac{b}{a}}\right) .$$

The function $v(t)$ whose Laplace transform is $V(s)$, then is

$$v(t) = \mathcal{L}^{-1}[V(s)] = \frac{1}{b}\left(\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+\frac{b}{a}}\right]\right) = \frac{1}{b}\left(1 - e^{-\frac{b}{a}t}\right) \,,$$

for $t > 0$ and $v(t) = 0$ for $t < 0$. This is the unit step response, it is continuous at $t = 0$, i.e. $v(0-) = v(0+) = 0$.

We check that in the t-space

$$w(t) = \dot{v}(t)$$

and in the s-space

$$\frac{1}{s}W(s) = V(s) \,.$$

For $a = 2$ and $b = 4$ we get

$$w(t) = \frac{1}{2}e^{-2t} \,, \qquad W(s) = \frac{1}{2s+4} \,,$$

and

$$v(t) = \frac{1}{4}\left(1 - e^{-2t}\right) \,, \qquad V(s) = \frac{1}{s(2s+4)} \,.$$

3. Solve $2\dot{x} + 4x = e^{-t}$ with initial condition $x(0+) = 1$ using the Laplace transform.

Ans. We take the Laplace transform of the equation $2\dot{x} + 4x = e^{-t}$. On the LHS we use the t-derivative rule and obtain

$$\mathcal{L}\Big[2\dot{x}(t) + 4x(t)\Big] = 2\Big(s\,X(s) - x(0+)\Big) + 4X(s) = 2(s+2)\,X(s) - 2$$

where we have used $x(0+) = 1$, and on the RHS we obtain

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1} \,.$$

Thus, the Laplace transform of the ODE with initial condition is

$$2(s+2)\,X(s) - 2 = \frac{1}{s+1}$$

or

$$X(s) = \frac{1}{2}\left(\frac{2}{s+2} + \frac{1}{(s+1)(s+2)}\right) .$$

To do the inverse Laplace transform, we have to do a partial fraction decomposition. This means that we want to write

$$\frac{1}{(s+1)(s+2)} = \frac{u}{s+1} + \frac{v}{s+2} ,$$

and then determine u and v. Multiplying through with $s+1$ and then setting $s = -1$ we obtain $1 = u$. Similarly, to determine v we multiply through with $s+2$ and then set $s = -2$ and get $-1 = v$. Thus, we can write

$$X(s) = \frac{1}{2}\left(\frac{1}{s+1} + \frac{1}{s+2}\right) .$$

The function $x(t)$ whose Laplace transform is $X(s)$ then is

$$x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{2}\left(\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+2}\right]\right) .$$

Using the s-shift rule we obtain for $x(t)$

$$x(t) = \frac{1}{2}\left(e^{-t} + e^{-2t}\right) .$$

4. Solve $2\dot{x} + 4x = e^{-t}\cos(2t)$ with $x(0+) = 0$ using the Laplace transform.

Ans. We take the Laplace transform of the equation $2\dot{x} + 4x = e^{-t}\cos(2t)$. On the LHS we use the t-derivative rule and obtain

$$\mathcal{L}\Big[2\dot{x}(t) + 4x(t)\Big] = 2\Big(s\,X(s) - x(0+)\Big) + 4X(s) = 2(s+2)\,X(s)$$

where we have used $x(0+) = 0$, and on the RHS we use the s-shift rule and obtain

$$\mathcal{L}[e^{-t}\cos(2t)] = \frac{s+1}{(s+1)^2+4} .$$

Thus, the Laplace transform of the ODE with initial condition is

$$2(s+2)\,X(s) = \frac{s+1}{(s+1)^2+4}$$

or

$$X(s) = \frac{1}{2}\frac{s+1}{(s+2)[(s+1)^2+4]} .$$

To do the inverse Laplace transform, we have to do a partial fraction decomposition. This means that we want to write

$$\frac{s+1}{(s+2)[(s+1)^2+4]} = \frac{u}{s+2} + \frac{v(s+1)+w}{(s+1)^2+4} ,$$

and then determine u, v, and w. Multiplying through with $s+2$ and then setting $s=-2$ we obtain $-1/5 = u$. Similarly, to determine v and w we multiply through with $(s+1)^2+4$ and then set $s=-1+2i$ since for $s=-1+2i$ the term $(s+1)^2+4$ equals zero. We get

$$\frac{2i}{1+2i} = \frac{2i(1-2i)}{5} = \frac{2i+4}{5} = 2iv + w \; .$$

We find $v = 1/5$ and $w = 4/5$. Thus, we can write

$$X(s) = \frac{1}{10}\left(-\frac{1}{s+1} + \frac{s+1}{(s+1)^2+4} + \frac{4}{(s+1)^2+4}\right) \; .$$

The function $x(t)$ whose Laplace transform is $X(s)$ then is

$$x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{10}\Big(-\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+4}\right] + 2\,\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+4}\right]\Big) \; .$$

Using the s-shift rule we can write

$$x(t) = \frac{1}{10}\Big(-e^{-t} + e^{-t}\cos(2t) + 2\,e^{-t}\sin(2t)\Big) \; .$$