18.03 Recitation 19, April 25, 2006

Hour exam review

Solution suggestions

Suppose you have an LTI system (what does that term mean?) which is modeled by a differential operator $p(D)$. You don't know the coefficients of the operator, but you investigate the system by delivering a blow to it—a unit impulse input signal—and recording the system response, $w(t)$. (Of course, it was at rest beforehand.)

Ans. The defining properties of any LTI system are, **linearity** and **time invariance**:

Linearity means that the relationship between the input and the output of the system satisfies the scaling and superposition properties. If the input of the system is

$$q(t) = Aq_1(t) + Bq_2(t)\,,$$

then the output of the system will be

$$x(t) = Ax_1(t) + Bx_2(t)\,,$$

for any constants A and B. $x_i(t)$ is the response when the input is $q_i(t)$.

Time invariance means that whether we apply an input to the system now or t seconds from now, the output will be identical, except a time delay of the t seconds.

Since our system is modeled by the differential operator $p(D)$ we have

$$p(D)x(t) = q(t)\ .$$

Review what we know from this:

(1) How do we determine the characteristic polynomial (and hence the coefficients) of the operator?

Ans. The Laplace transform $W(s) = \mathcal{L}[w(t)]$ of the unit impulse response satifies

$$W(s) = \frac{1}{p(s)}\ .$$

Thus, from $W(s)$ we can find out the polynomial defining $p(D)$. In the example we have

$$\mathcal{L}[w(t)] = \mathcal{L}\Big[e^{-t}\sin(3t)\Big] = \frac{3}{(s+1)^2+9} = \frac{3}{s^2+2s+10}\ .$$

Thus,

$$p(s) = \frac{1}{3}s^2 + \frac{2}{3}s + \frac{10}{3}\ .$$

(2) How can we write down (in terms of $w(t)$) the solution (with rest initial conditions) to $p(D)x = q(t)$ for some arbitrary input signal?

Ans. The solution of $p(D)x = q(t)$ with rest initial conditions is given by the convolution, i.e.

$$x(t) = w(t) * q(t) = \int_0^t w(\tau)\,q(t-\tau)\,d\tau = \int_0^t w(t-\tau)\,q(\tau)\,d\tau\ .$$

In our example we have

$$x(t) = \int_0^t e^{-\tau} \sin(3\tau)\, q(t-\tau)\, d\tau \ .$$

(3) How can we determine the multiplier $W(r)$ such that $x_p = W(r)e^{rt}$ is a system response to the exponential input signal e^{rt} (for r constant)?

Ans. By the exponential response formula (c.f. Recitation 10) the response of the system to an input signal e^{rt} is given by $\frac{1}{p(r)}e^{rt}$ (as long as $p(r) \neq 0$ otherwise we have to apply the resonant ERF). We already know from **(2)** how to compute $\frac{1}{p(r)}$, it is $W(r)$ where W is the Laplace transform of the unit impulse response $w(t)$. This means

$$x_p(t) = \frac{1}{p(r)}e^{rt} = W(r)e^{rt} \ .$$

In our example we have

$$x_p(t) = \frac{1}{p(r)}e^{rt} = \frac{3}{r^2+2r+10}e^{rt} \ .$$

(4) How can we determine the frequency response of the system?—that is, A and ϕ (both functions of ω) such that $p(D)x = \cos(\omega t)$ has sinusoidal solution $A\cos(\omega t - \phi)$?

Ans. We complexify the differential equation and obtain

$$p(D)z(t) = e^{i\omega t} \ ,$$

and remember that $x(t) = \operatorname{Re} z(t)$. The solution to the complex ODE is again given by the exponential response formula. It is

$$z_p(t) = \frac{1}{p(i\omega)}e^{i\omega t}$$

as long as $p(i\omega) \neq 0$ (otherwise the solution wouldn't be periodic). If we write the complex number $p(i\omega)$ as

$$p(i\omega) = |p(i\omega)|e^{i\phi} \ ,$$

then $z_p(t)$ becomes

$$z_p(t) = \frac{1}{|p(i\omega)|}e^{i(\omega t - \phi)} \ .$$

And the real part of it is

$$x_p(t) = \operatorname{Re} z_p(t) = \frac{1}{|p(i\omega)|}\cos(\omega t - \phi) \ .$$

So we have $A = \frac{1}{|p(i\omega)|}$ and ϕ is the phase of the complex number $p(i\omega)$. Since $\frac{1}{p(s)}$ is $W(s)$ we have

$$A = \frac{1}{|p(i\omega)|} = \left|\frac{1}{p(i\omega)}\right| = \left|W(i\omega)\right| \ ,$$

and ϕ is such that

$$W(i\omega) = \left|W(i\omega)\right| e^{-i\phi} \ .$$

In the example we have

$$W(s) = \frac{3}{s^2 + 2s + 10} \ ,$$

and therefore

$$W(i\omega) = \frac{3}{(10 - \omega^2) + 2i\omega} = \frac{2}{\sqrt{(10-\omega^2)^2 + 4\omega^2}} e^{-i\phi}$$

where ϕ satisfies

$$\tan\phi = \frac{2\omega}{10 - \omega^2} \ .$$

A unifying visual image is the graph of $|W(s)|$, which is largely controlled by the poles of $W(s)$.

Ans. In our example we have

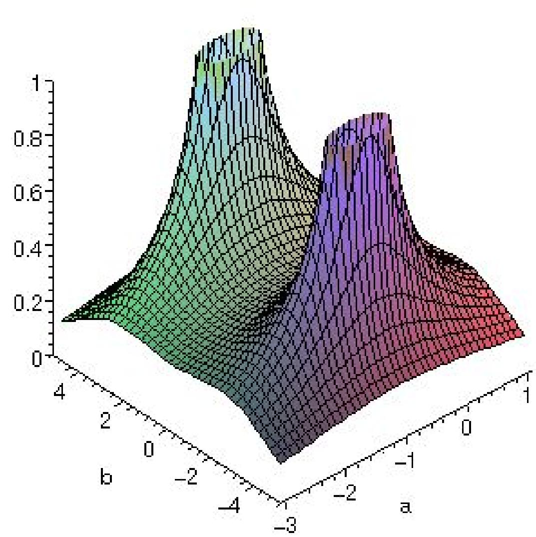
$$W(s) = \frac{3}{(s + 1 + 3i)(s + 1 - 3i)} \ ,$$

and we have poles of $W(s)$ and thus of $|W(s)|$ at $s = -1 \pm 3i$. In terms of $p(s)$, this means that $p(s)$ has zeros at these poles. From the resonant response formula we know that this happens if we hit the resonance frequency of our system.

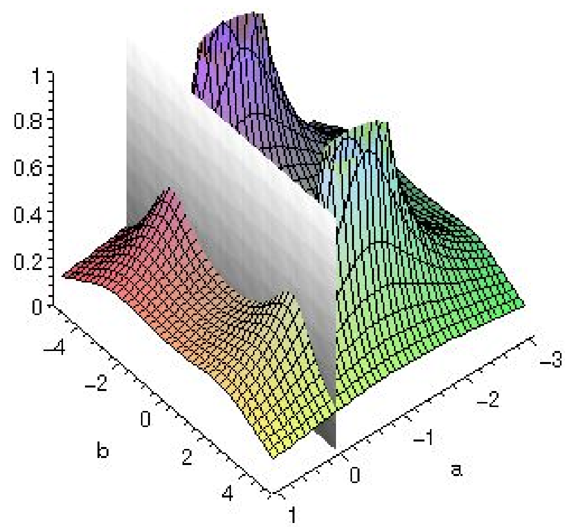
In our example, we have

$$\Big|W(s)\Big| = \frac{3}{\Big|(s+1)^2 + 9\Big|} \ .$$

The graph of $|W(s)|$ for $s = a + ib$ looks like this:



The amplitude response curve, $|W(i\omega)|$, fits into this graph: as its intersection of the vertical plane over the imaginary axis. In the picture below this is the vertical gray plane. The near resonance occurs as you rise up on the shoulder of the mountain rising to the pole:



This leads to the following picture of the amplitude response curve $|W(i\omega)|$: