18.03 Recitation 20, April 27, 2006

Systems of first order equations

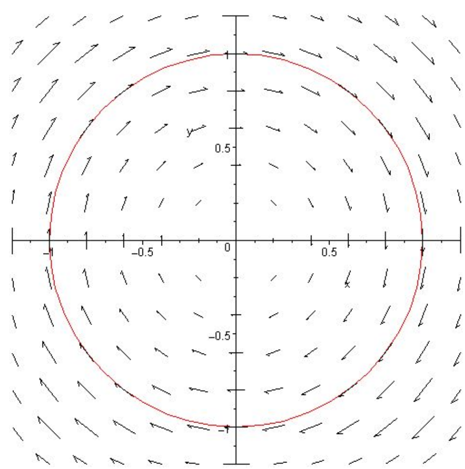
Solution suggestions

1. Define a vector field in the plane by putting the vector $y\mathbf{i} - x\mathbf{j}$ at the position (x, y). Sketch enough values of this vector field to visualize it and describe it in words. Then sketch a curve which is everywhere tangent to it and passes though the point $(1, 0)$.

Ans. We observe that the vector $y\mathbf{i} - x\mathbf{j}$ at the position (x, y) is always perpendicular to the vector $x\mathbf{i} + y\mathbf{j}$ pointing from the orgin to that position. Moreover, the length of the vector is

$$||x\mathbf{i} + y\mathbf{j}|| = \sqrt{x^2 + y^2} \ .$$

This means that we can imagine this vector field as follows: at all points of a concentric circle of radius r about the origin, i.e. points with coordinates (x, y) such that $x^2 + y^2 = r^2$, the vector has length r and is tangent to the circle. We also see that at the poistion $(1, 0)$ the vector is $-\mathbf{j}$, and at $(0, 1)$ it's $\mathbf{i}$. In general we see that the vector field is tangent to the circle and pointing counterclockwise. We get the following picture:



The curve you drew is the trajectory of a solution of the system of ODEs

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

Solve this system of equations in the following way: substitute $\dot{y} = -x$ into the equation you get for $\ddot{x}$ by differentiating $\dot{x} = y$. This gives you a second order LTI ODE. The initial conditions for x and y give initial conditions for this new ODE. Solve it.

Ans. We start out with

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

and the initial conditions $x(0) = 1$ and $y(0) = 0$ as the curve is passing through the point $(1, 0)$. Now, we take the first equation $\dot{x} = y$ and differentiate it to obtain

$$\ddot{x} = \dot{y} \; .$$

But from the second equation we know $\dot{y} = -x$. Substituting leads to

$$\ddot{x} + x = 0 \; .$$

This means that instead of having two (coupled) first order ODEs we now have one second order ODE in x. The general solution of it is

$$x(t) = A \cos t + B \sin t \; .$$

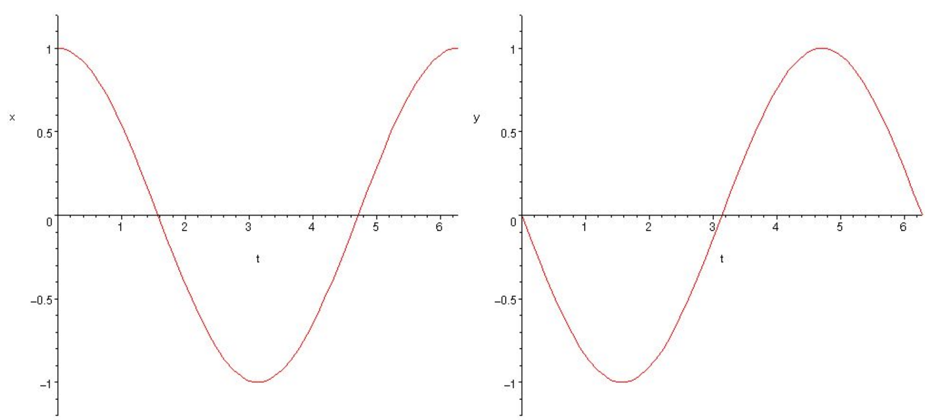
What about the initial conditions? We know that we need two initial conditions to determine the solution of a second order linear ODE. Clearly $x(0) = 0$ is one. But as $\dot{x} = y$ we have in particular $\dot{(}0)x = y(0) = 0$. Therefore, our initial conditions are $x(0) = 1$ and $\dot{x}(0) = 0$. Therefore, $A = 1$ and $B = 0$. Thus, we have determined $x(t) = \cos t$.

Now, how do we get y? We have the equation $\dot{x} = y$ – thus, we find

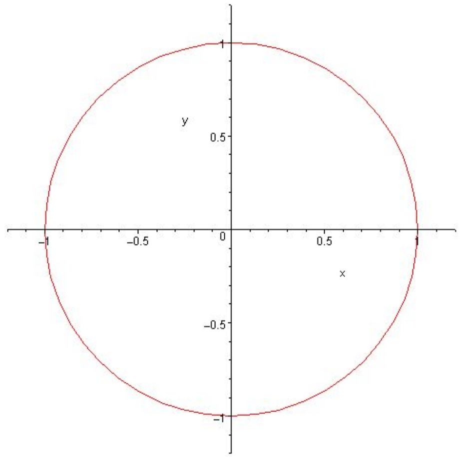
$$x(t) = \cos t \; , \qquad y(t) = -\sin t \; .$$

Then graph x against t, graph y against t, and plot the path of the curve $\mathbf{u}(t)$ in the plane. Does it look right?

Ans. Here are the graphs of $x(t)$ and $y(t)$ plotted against t:



And here is $y(t)$ plotted against $x(t)$:



2. Now reverse engineer this, starting with the second order IVP

$$\ddot{x} + (1/2)\dot{x} + (17/16)x = 0,$$

with initial condition $x(0) = 1$, $\dot{x}(0) = 0$. Use $y = \dot{x}$ for one of the pair of equations. So: write an equation for $\dot{y}$ in terms of x and y. Together this pair of equations determines $\dot{\mathbf{u}}$ in terms of $\mathbf{u}$. Solve the original second order ODE, and reinterpret your solution as a solution of the system you produced. Sketch graphs of x and of y as functions of t, and sketch the path of the curve $\mathbf{u}(t)$ (its "trajectory"). If you think of the variable x in the original equation as position, how is velocity, $\dot{x}$, represented in the picture of the trajectory?

Ans. If we take $y = \dot{x}$ then we get the equation

$$\dot{y} = \ddot{x} = -\frac{1}{2}\dot{x} - \frac{17}{16}x = -\frac{1}{2}y - \frac{17}{16}x \ ,$$

that is

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{1}{2}y - \frac{17}{16}x \end{cases} \tag{1}$$

with initial conditions $x(0) = 1$ and $y(0) = 0$. If we think of the vector $\mathbf{u}(t)$ having the components $x(t)$ and $y(t)$ we can write

$$\mathbf{u}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} .$$

The velocity vector $\dot{\mathbf{u}}$ then is

$$\dot{\mathbf{u}}(t) = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} .$$

This means we can write Eq. (1) as

$$\begin{aligned} \dot{\mathbf{u}}(t) &= \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{17}{16} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -\frac{17}{16} & -\frac{1}{2} \end{bmatrix} \mathbf{u}(t) . \end{aligned}$$

Now, we solve the original second order ODE: the characteristic polynomial is $p(s) = s^2 + \frac{1}{2}s + \frac{17}{16}$ which has roots

$$s_\pm = -\frac{1}{4} \pm \sqrt{\frac{1}{16} - \frac{17}{16}} = -\frac{1}{4} \pm i \,.$$

So, the general solution has the form

$$x(t) = e^{-\frac{t}{4}}\Big(A\cos(t) + B\sin(t)\Big) \,.$$

The initial condition $x(0) = 1$ gives $A = 1$. Taking the derivative we obtain

$$\dot{x}(t) = e^{-\frac{t}{4}}\Big([B - \frac{1}{4}]\cos(t) + [-1 - \frac{B}{4}]\sin(t)\Big) \,.$$

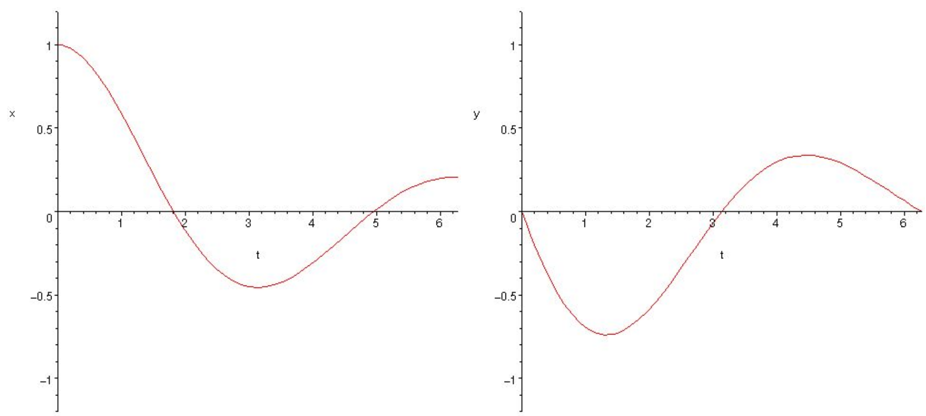
Thus $\dot{x}(0) = 0$ gives $B = \frac{1}{4}$ and we have found the solution

$$x(t) = e^{-\frac{t}{4}}\Big(\cos(t) + \frac{1}{4}\sin(t)\Big) \,,$$

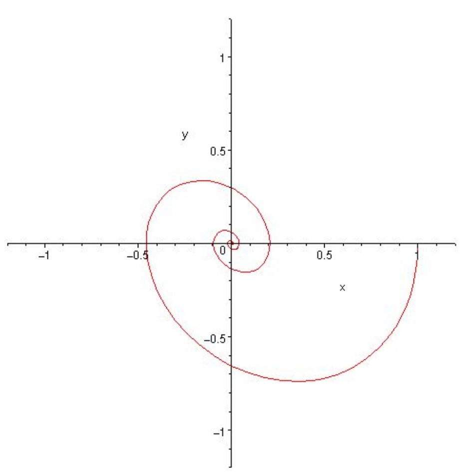
and

$$y(t) = -\frac{17}{16}e^{-\frac{t}{4}}\sin(t) \,,$$

Here are the graphs of $x(t)$ and $y(t)$ plotted against t:



And here is $y(t)$ plotted against $x(t)$, the so called trajectory:



3. Practice in matrix multiplication: Compute the following products:

$$\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} x & y \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & u \\ y & v \end{bmatrix}.$$

Ans. A (n, m)-matrix is a matrix with n rows and m columns. A vector with two components can also be regarded as a $(2, 1)$-matrix. Remember, that when you multiply an (n, m)-matrix with an (m, k) matrix you obtain a (n, k)-matrix. We compute the following products: A $(1, 2)$-matrix times a $(2, 1)$-matrix should give a $(1, 1)$-matrix which is just a number:

$$\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1\,x + 2\,y = x + 2y\ .$$

A $(2, 1)$-matrix times a $(1, 2)$-matrix gives a $(2, 2)$-matrix:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x & y \\ 2x & 2y \end{bmatrix}\ .$$

A $(2, 2)$-matrix times a $(2, 1)$-matrix (a vector) gives a $(2, 1)$-matrix (another vector):

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}\ .$$

A $(2, 2)$-matrix times a $(2, 2)$-matrix gives a $(2, 2)$-matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & u \\ y & v \end{bmatrix} = \begin{bmatrix} ax + by & au + bv \\ cx + dy & cu + dv \end{bmatrix}\ .$$