18.03 Recitation 23, May 9, 2006

Qualitative analysis of linear systems

Solutions suggestions

The matrices I want you to study all have the form $A = \begin{bmatrix} a & 2 \\ -2 & -1 \end{bmatrix}$.

1. Compute the trace, determinant, characteristic polynomial, and eigenvalues, in terms of a.

Ans. The trace is

$$\mathrm{tr}\, A = a - 1 \,,$$

and the determinant is

$$\det A = a(-1) - 2(-2) = -a + 4 \,.$$

Thus, the characteristic polynomial is

$$p_A(\lambda) = \det\Big(A - \lambda I\Big) = \lambda^2 - \mathrm{tr}\,(A)\,\lambda + \det(A) = \lambda^2 - (a-1)\,\lambda - a + 4 \,.$$

The eigenvalues are the roots of the characteristic polynomial. The roots are

$$\lambda = \frac{a-1}{2} \pm \sqrt{\left(\frac{a-1}{2}\right)^2 + a - 4} = \frac{a-1}{2} \pm \frac{1}{2}\sqrt{(a-1)^2 + 4a - 16} \,.$$

2. For these matrices, express the determinant as a function of the trace. Sketch the $(\mathrm{tr}\ A, \det\ A)$ plane, along with the critical parabola $\det\ A = (\mathrm{tr}\ A)^2/4$, and plot the curve representing the relationship you found for this family of matrices. On this curve, plot the points corresponding to the following values of a: $a = -6, -5, -2, 1, 2, 3, 4, 5$.

Ans. From **(1)** we see

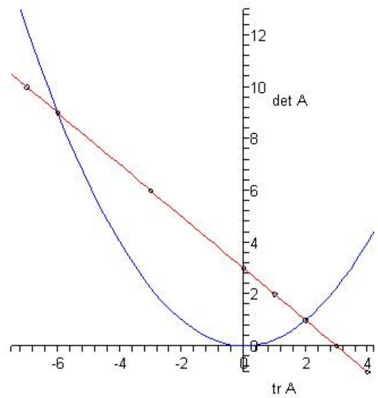
$$\det(A) = -a + 4 = -(a-1) + 3 = -\mathrm{tr}\,(A) + 3 \,.$$

We notice that if the matrix lies on the critical parabola $\det A = (\mathrm{tr}\ A)^2/4$, then the charcateristic polynomial is

$$p_A(\lambda) = \lambda^2 - \mathrm{tr}\,(A)\,\lambda + \det(A) = \lambda^2 - \mathrm{tr}\,(A)\,\lambda + \left(\frac{\mathrm{tr}\,A}{2}\right)^2 = \left(\lambda - \frac{\mathrm{tr}\,A}{2}\right)^2 \,.$$

This means that for matrices which lie on the critical parabola we have only one (repeated) eigenvalue $\mathrm{tr}\,A/2$.

Here is the sketch of the $(\mathrm{tr}\ A, \det\ A)$ plane:



3. Make a table showing for each a in this list (1) the eigenvalues; (2) information about the phase portrait derived from the eigenvalues (real and distinct; real and repeated; non-real) and the stability type (stable if all real parts are negative; unstable if at least one real part is positive; undesignated if neither); (3) further information beyond what the eigenvalues alone tell you: if a spiral, the direction (clockwise or counterclockwise) of motion; if the eigenvalues are repeated, whether the matrix is defective or complete.

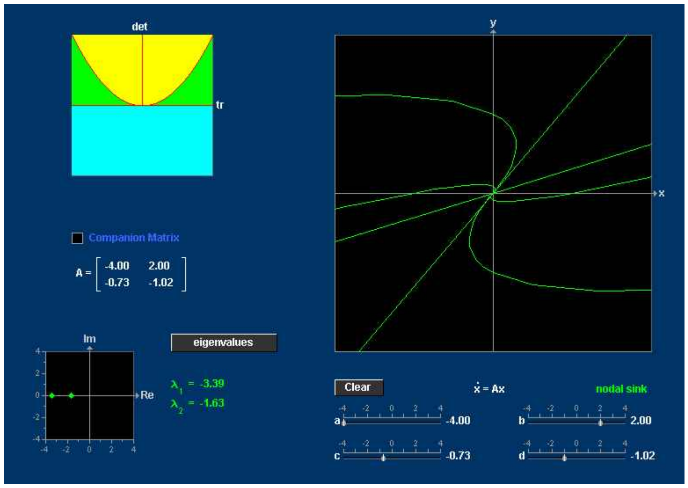
Ans.

a	$(\text{tr}\,(A), \det A)$	eigenvals	phase portrait	stability	further info
-6	$(-7, 10)$	$-5, -2$	real, distinct	stable	node
-5	$(-6, 9)$	-3	real, repeated	stable	defective node
-2	$(-3, 6)$	$-\frac{3}{2} \pm \frac{i}{2}\sqrt{15}$	complex conjugate	stable	spiral (clockwise)
1	$(0, 3)$	$\pm i\sqrt{3}$	purely imaginary	neutrally stable	center (clockwise)
2	$(1, 2)$	$\frac{1}{2} \pm \frac{i}{2}\sqrt{7}$	complex conjugate	unstable	spiral (clockwise)
3	$(2, 1)$	1	real, repeated	unstable	defective node
4	$(3, 0)$	$0, 3$	real, distinct (one zero)	unstable	degenerate comb
5	$(4, -1)$	$2 \pm \sqrt{5}$	real, distinct (opposite sign)	unstable	saddle

(i), $a = -6$**:** The eigenvalues are *real* and of the same sign, but *distinct*, You have a *node*. Both normal modes decay to zero, Thus, it's *asymptotically stable*: all solutions $\to 0$ as $t \to \infty$. But the one with eigenvalue -5 decays much faster: so the non-normal mode trajectories become tangent to this eigenline. The general solution is

$$c_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} .$$

Here is a picture of a similar pase portrait (the exact parameters are out of reach in the Mathlet):



(ii), $a = -5$: This matrix lies on the critical parabola. We have a repeated real eigenvalue -3. It is *asymptotically stable*. Computing the eigenvectors one finds only one eigenvector. Thus, it's a *defective node*. If one computes the eigenvector one finds

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

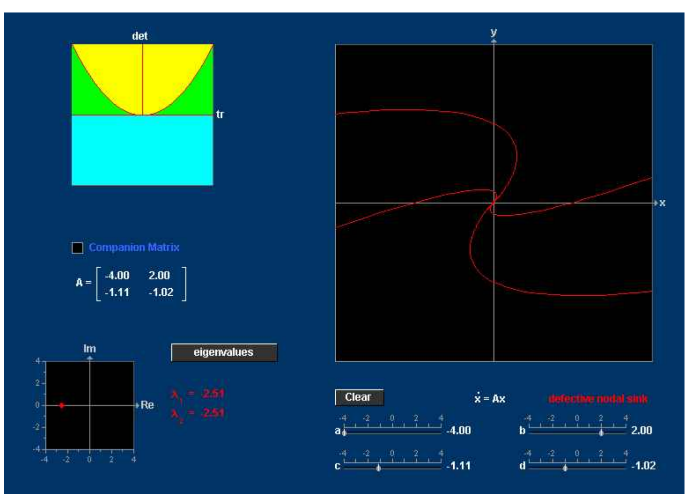
Solving $(A + 3I)\mathbf{w} = \mathbf{v}$ we find

$$\mathbf{w} = \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} ,$$

Thus, the general solution is

$$c_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \right) .$$

Here is a picture of the pase portrait (the exact parameters are out of reach in the Mathlet):



(iii), $a = -2$: We have two complex eigenvalues. In fact, they are *complex conjugates* of each other. This is a *spiral*. The spirals *move in* as we have $\text{Re}\,(\lambda) < 0$. This means that it's *stable*. To determine which way they move in we determine $\dot{\mathbf{u}}$ when $\mathbf{u} = [1; 0]$. We compute

$$\dot{\mathbf{u}} = A\mathbf{u} = \begin{bmatrix} -2 \\ -2 \end{bmatrix},$$

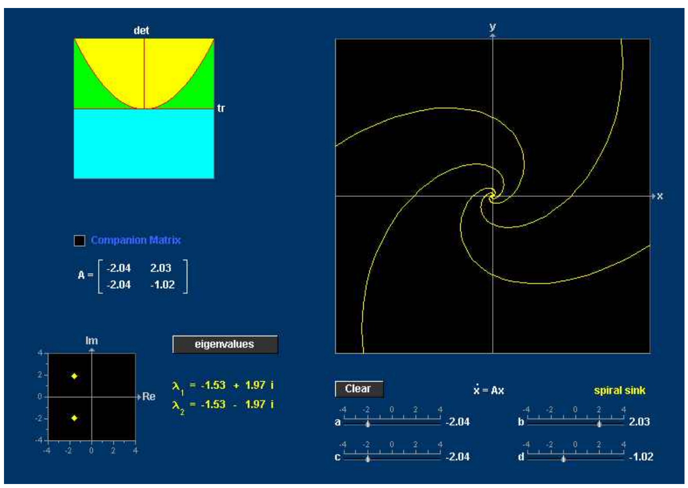
which is pointing to the left and down. Thus, the spirals are moving in *clockwise*. The eigenvector for $-\frac{3}{2} + \frac{i}{2}\sqrt{15}$ is

$$\mathbf{v} = \begin{bmatrix} 1 - i\sqrt{15} \\ 4 \end{bmatrix}.$$

Thus, the general solution is

$$\begin{aligned} & c_1 \quad e^{-\frac{3}{2}t} \left(\cos\left(\frac{\sqrt{15}}{2}t\right) \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \sin\left(\frac{\sqrt{15}}{2}t\right) \begin{bmatrix} \sqrt{15} \\ 0 \end{bmatrix} \right) \\ + & c_2 \quad e^{-\frac{3}{2}t} \left(-\cos\left(\frac{\sqrt{15}}{2}t\right) \begin{bmatrix} \sqrt{15} \\ 0 \end{bmatrix} + \sin\left(\frac{\sqrt{15}}{2}t\right) \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right) \end{aligned}$$

Here is a picture of the pase portrait:



(iv), $a = 1$: We have two *purely imaginary* eigenvalues. Thus, the trajectories are ellipses. The technical term for this type of phase portrait is *center*. Since all trajectories stay bounded (but they do NOT go to zero for $t \to \infty$) it's *neutrally stable*. To determine which way the ellipses turn we determine $\dot{\mathbf{u}}$ when $\mathbf{u} = [1; 0]$. We compute

$$\dot{\mathbf{u}} = A\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

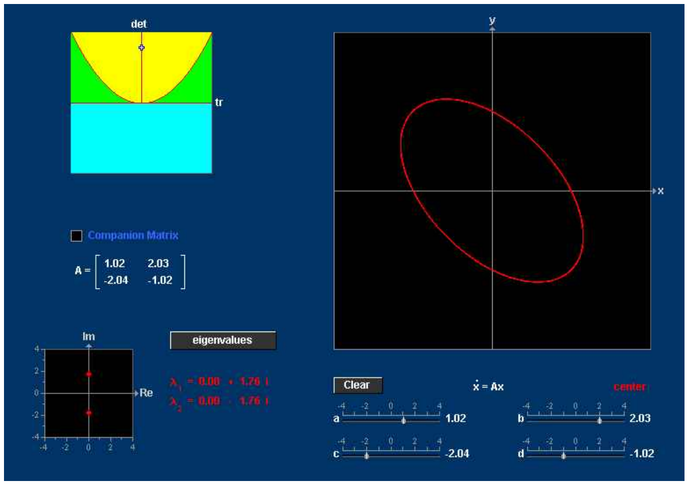
which is to the right and down. Thus, the ellipses are turning *clockwise*. The eigenvector for $i\sqrt{3}$ is

$$\mathbf{v} = \begin{bmatrix} -1 - i\sqrt{3} \\ 2 \end{bmatrix}.$$

Thus, the general solution is

$$\begin{aligned} & c_1 \quad \left(\cos\left(\sqrt{3}t\right) \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \sin\left(\sqrt{3}t\right) \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix} \right) \\ + & c_2 \quad \left(-\cos\left(\sqrt{3}t\right) \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix} + \sin\left(\sqrt{3}t\right) \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) \end{aligned}$$

Here is a picture of the pase portrait:



(v), $a = 2$**:** We have two complex eigenvalues. In fact, they are *complex conjugates* of each other. This is a *spiral*. The spirals *move out* as we have $\text{Re}(\lambda) > 0$. This means that it's *unstable*. To determine which way they moves out we determine $\dot{\mathbf{u}}$ when $\mathbf{u} = [1; 0]$. We compute

$$\dot{\mathbf{u}} = A\mathbf{u} = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

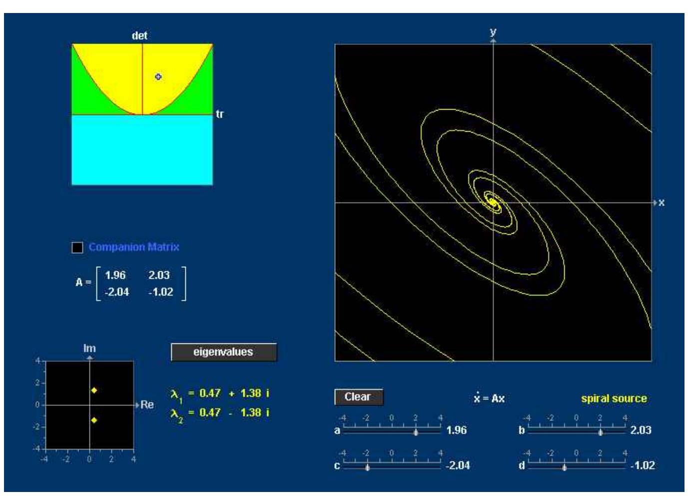
which is pointing right and down. Thus, the spirals are moving out *clockwise*. The eigenvector for $\frac{1}{2} + \frac{i}{2}\sqrt{7}$ is

$$\mathbf{v} = \begin{bmatrix} 4 \\ -3 + i\sqrt{7} \end{bmatrix}.$$

Thus, the general solution is

$$\begin{aligned} & c_1 \quad e^{\frac{1}{2}t} \left(\cos\left(\frac{\sqrt{7}}{2}t\right) \begin{bmatrix} 4 \\ -3 \end{bmatrix} - \sin\left(\frac{\sqrt{7}}{2}t\right) \begin{bmatrix} \sqrt{7} \\ 0 \end{bmatrix} \right) \\ + & c_2 \quad e^{\frac{1}{2}t} \left(\cos\left(\frac{\sqrt{7}}{2}t\right) \begin{bmatrix} \sqrt{7} \\ 0 \end{bmatrix} + \sin\left(\frac{\sqrt{7}}{2}t\right) \begin{bmatrix} 4 \\ -3 \end{bmatrix} \right) \end{aligned}$$

Here is a picture of the pase portrait:



(vi), $a = 3$: This matrix lies again on the critical parabola. We have a repeated real eigenvalue 1. It is *asymptotically unstable*. Computing the eigenvectors one finds only one eigenvector. Thus, it's a *defective node*. If one computes the eigenvector one finds

$$\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

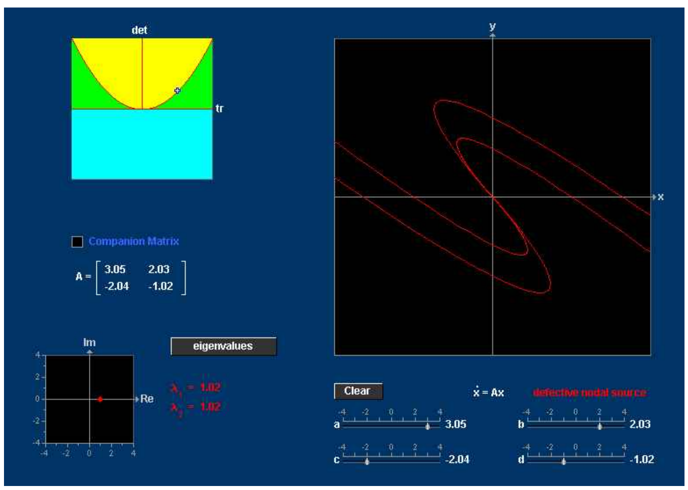
Solving $(A - I)\mathbf{w} = \mathbf{v}$ we find

$$\mathbf{w} = \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}.$$

Thus, the general solution is

$$c_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^t \left(t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \right).$$

Here is a picture of the pase portrait:



(vii), $a = 4$: We have $\det A = 0$. That is the *degenerate* case. One of the eigenvalues is zero. The other is positive. Thus, it's *asymptotically unstable*. In fact, it's an *unstable comb*.

The eigenvector $\mathbf{v}_1$ corresponding to $\lambda_1 = 0$ is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

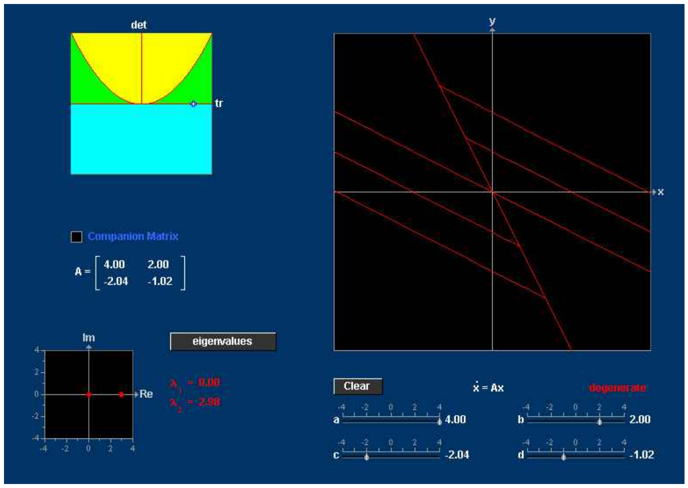
The eigenvector $\mathbf{v}_2$ corresponding to $\lambda_2 = 3$ is

$$\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Thus, the general solution is

$$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

For $c_2 = 0$ there is a line (at least) of constant solutions. Here is a picture of the pase portrait:



(viii), $a = 5$: The eigenvalues are real and of opposite sign, the phase portrait is a *saddle*. There are two eigenlines, one with positive eigenvalue and the other with negative. Normal modes along one move out, and along the other move in. Thus, it's *unstable*.

The eigenvector $\mathbf{v}_1$ corresponding to $\lambda_1 = 2 + \sqrt{5}$ is

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 3 - \sqrt{5} \end{bmatrix}.$$

The eigenvector $\mathbf{v}_2$ corresponding to $\lambda_2 = 2 - \sqrt{5}$ is

$$\mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 + \sqrt{5} \end{bmatrix}.$$

Thus, the general solution is

$$c_1 e^{(2+\sqrt{5})t} \begin{bmatrix} -2 \\ 3 - \sqrt{5} \end{bmatrix} + c_2 e^{(2-\sqrt{5})t} \begin{bmatrix} -2 \\ 3 + \sqrt{5} \end{bmatrix}.$$

Here is a picture of the pase portrait (the exact parameters are out of reach in the Mathlet):