18.03 Recitation 25, May 16, 2006

Qualitative analysis of nonlinear systems

1. (a) Talk through an interpretation of the nonlinear autonomous system

$$\dot{x} = (6 - x - 2y)x$$

$$\dot{y} = (6 - y - 2x)y$$

in terms of two populations. What effect do they have on each other's growth rates?

(b) Find where this vector field is vertical and where it is horizontal and plot these curves (or lines). Find the critical points. Compute the Jacobian matrix

$$J(x, y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix}$$

For each critical point (a, b), compute $J(a, b)$. Is the critical point nondegenerate? If so, what is the linearization at it? If spiral, check stability and direction of time. If node or saddle, determine the eigenlines with their arrows of time, and if node, which eigenline solutions become tangent to as the near the critical point. Snap these little sketches in and complete to a phase portrait. If your recitation room has MIT server terminal in it, you may want to check your answer using MATLAB's `pplane6`.

(c) Finally, pick an initial condition and sketch graphs of the two functions $x(t)$ and $y(t)$. You won't be able to be very precise about the time direction, given the information presented by the phase portrait. When t is large, however, you should be able to write down what $x(t)$ and $y(t)$ are to a good approximation, using the linearization.

2. Find the general solution to the differential equation
$\dot{\mathbf{u}} = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} \mathbf{u} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Sketch the phase portrait.

3. Now use variation of parameters to find a particular solution to
$\dot{\mathbf{u}} = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} \mathbf{u} + \begin{bmatrix} t \\ 0 \end{bmatrix}$.
(Leaving the answer in the form of a matrix times a vector will be fine.)

Is there a phase portrait for this equation?

4. We saw how second order equations give rise to a "companion" first order system of equations, by means of the equation $y = \dot{x}$. Suppose $x(t) = 1 + \sin(2t)$. Sketch the trajectory of the vector-valued function $\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$.